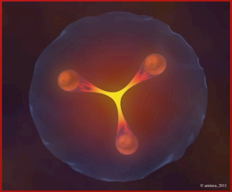
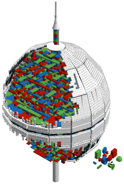
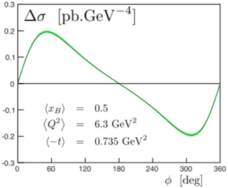
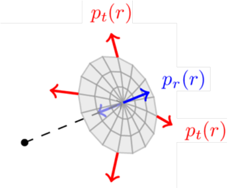
DE LA RECHERCHE À L'INDUSTRIE



Exclusive reactions as a nuclear manometer







EJC 2022 | Hervé MOUTARDE

Sep. 2022

université PARIS-SACLAY

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Manometer

A **manometer** is a device for measuring the pressure of gases and liquids.

✍ Cambridge dictionary (2022)

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Several questions

- A **fluid picture** of the nucleon?

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Several questions

- A **fluid picture** of the nucleon?
- **Internal pressure** of the nucleon?

Motivation.

Investigation of the energy-momentum structure of the nucleon.

Manometer

A **manometer** is a device for measuring the pressure of gases and liquids.

✍ Cambridge dictionary (2022)

Several questions

- A **fluid picture** of the nucleon?
- **Internal pressure** of the nucleon?
- **Exclusive reactions** as a measuring device?

Manometer

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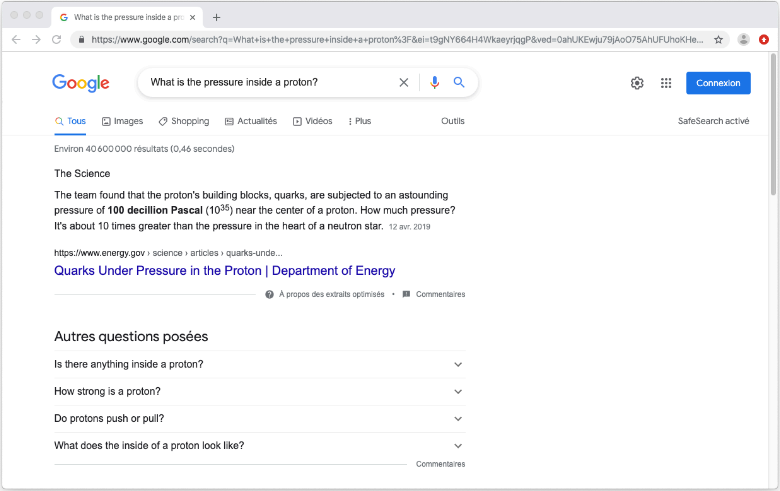
Several questions

- A **fluid picture** of the nucleon?
- **Internal pressure** of the nucleon?
- **Exclusive reactions** as a measuring device?

Can we talk about a **proton internal pressure** or other properties borrowed from fluid mechanics?

What is the proton internal pressure?

Clarify the concept by association of ideas.



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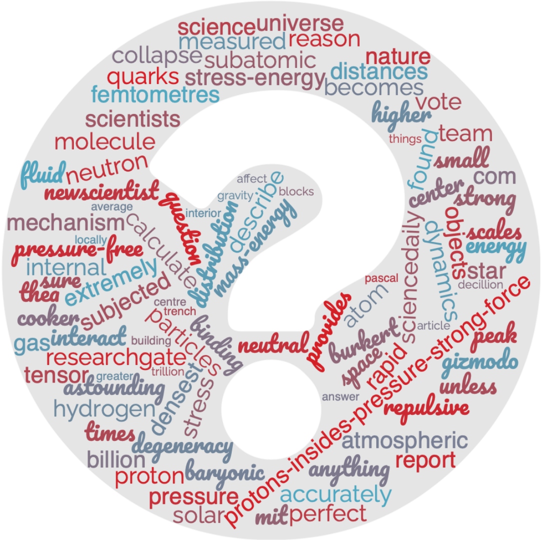
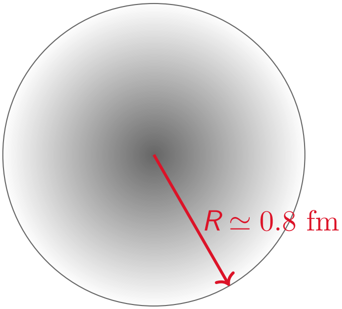
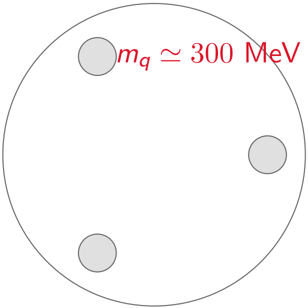


Keywords

- Fluid, gas
- Stress-energy
- Tensor
- ...



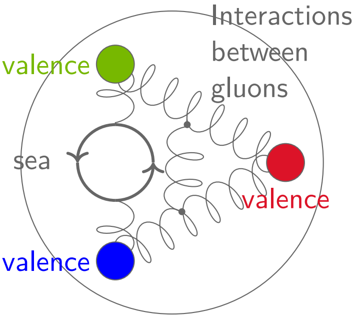
- **Composite** object with an **electric charge** spread over a spherical region.



- **Composite** object with an **electric charge** spread over a spherical region.
- Quark model description: **nonrelativistic** bound state of **3 massive quarks**.

Hadrons and partons.

The nucleon: a **quantum relativistic** system of **confined** particles.



- **Composite** object with an **electric charge** spread over a spherical region.
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- Modern description (QCD): **relativistic** bound state of **colored light** quarks and **massless** gluons (**partons**).

Hadrons and partons.

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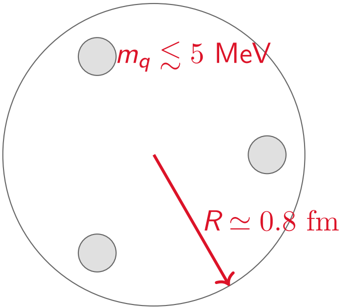
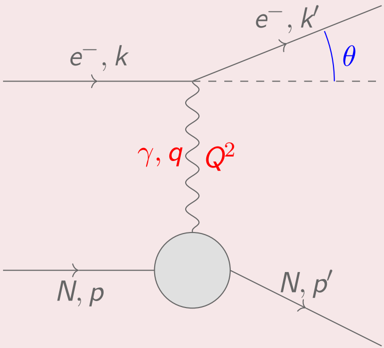


- **Composite** object with an **electric charge** spread over a spherical region.
- Quark model description: **nonrelativistic** bound state of **3 massive quarks**.
- Modern description (QCD): **relativistic** bound state of **colored light** quarks and **massless** gluons (**partons**).
- **Electric charge radius** $\simeq$ 0.8 fm.
- Need for a **quantum relativistic** framework:
  - Uncertainty principle $\Delta p \simeq$ 350 MeV.
  - Electric charge radius $\simeq 4\times$ Compton wavelength.

Kinematics of elastic scattering on the nucleon



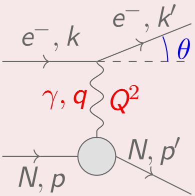
$$
\begin{aligned}
q &\equiv k - k', \\
Q^2 &\equiv -q^2, \\
\nu &\equiv \frac{p \cdot q}{M}, \\
x_B &\equiv \frac{Q^2}{2p \cdot q}, \\
W^2 &\equiv (p+q)^2, \\
s &\equiv (p+k)^2.
\end{aligned}
$$

In the target rest frame $\theta \in [0, \pi]$ and:

$$p \equiv (M, \vec{0}), \quad k \equiv (E, \vec{k}), \quad k' \equiv (E', \vec{k}').$$

Kinematics of elastic scattering on the nucleon

$$q \equiv k - k', \quad Q^2 \equiv -q^2, \quad \nu \equiv \frac{p \cdot q}{M},$$



$$x_B \equiv \frac{Q^2}{2p \cdot q}, \quad W^2 \equiv (p+q)^2, \quad s \equiv (p+k)^2.$$

Exercise 0.1

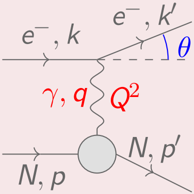
Give the typical energy range to probe the nucleon structure with electromagnetic elastic scattering. Justify the neglect of the electron mass and show that $q^2 \simeq -4EE' \sin^2 \theta/2$ and $Q^2 > 0$.

- Electromagnetic current:

$$J^{\text{em}}_\mu(y) = \sum_{q=u,d,s,\ldots} e_q \bar{q}(y)\gamma_\mu q(y)$$

- From invariance under translations, take J^{em} at 0.

Kinematics of elastic scattering on the nucleon



- Amplitude $\mathcal{M}(eN \to eN)$ at **Born order**:

$$\mathcal{M}(eN \to eN) = \bar{u}(k', \lambda')\gamma^\mu u(k,\lambda)\frac{e^2}{q^2}\left\langle N, p', h' \left| J^{\text{em}}_\mu(0) \right| N, p, h \right\rangle$$

- Most general **Lorentz structure** ($q = p' - p$):

$$\left\langle \pi, p' \left| J_\mu^{\mathrm{em}}(0) \right| \pi, p \right\rangle = a_1 p_\mu + a_2 p'_\mu + a_3 q_\mu$$

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- Use **4-momentum conservation**:

$$\left\langle \pi, p' \left| J_\mu^{\mathrm{em}}(0) \right| \pi, p \right\rangle = (a_1 + a_2)\frac{(p+p')_\mu}{2} + \left(a_3 - \frac{a_1}{2} + \frac{a_2}{2} \right) q_\mu$$

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- Enforce **current conservation** $q^\mu J_\mu^{\mathrm{e.m.}} = 0$ with $q^2 < 0$:

$$0 = \left(a_3 - \frac{a_1}{2} + \frac{a_2}{2} \right) q^2$$

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- **Hermiticity** of $J^{\mathrm{e.m.}}_\mu$: there exist one **real** coefficient F such that:

$$\left\langle \pi, p' \left| J^{\mathrm{e.m.}}_\mu(0) \right| \pi, p \right\rangle = F (p + p')_\mu$$

Unpolarized elastic scattering at Born order.

Parameterization of the matrix element: spin-0 case.

- Most general **Lorentz structure** ($q = p' - p$):

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- F is **dimensionless** ($|\pi, p\rangle$ has mass dimension -1).
- F **depends on** q^2 **only** (elastic scattering: $-q^2 = 2p \cdot q$).

Unpolarized elastic scattering at Born order.

Parameterization of the matrix element: spin-1/2 case (1/2).

- Most general **Lorentz structure** ($q = p' - p$):

$$\left\langle N, p' \left| J_\mu^{\mathrm{em}}(0) \right| N, p \right\rangle = \bar{u}(p')\Gamma_\mu(p', p)u(p)$$

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- Expand Γ_μ in **16 matrices** 1, γ_ρ, $[\gamma_\rho, \gamma_\sigma]$, $\gamma_5\gamma_\rho$ and γ_5:

$$\begin{array}{lcl}
1 & : & p_\mu, p'_\mu \\
\gamma_\rho & : & \gamma_\mu, p_\mu \not{p}, p'_\mu \not{p}, p_\mu \not{p}', p'_\mu \not{p}' \\
[\gamma_\rho, \gamma_\sigma] & : & [\gamma_\mu, \not{p}], [\gamma_\mu, \not{p}'], [\not{p}, \not{p}'] p_\mu, [\not{p}, \not{p}'] p'_\mu \\
\gamma_5\gamma_\rho & : & \gamma_5\gamma_\rho \epsilon^{\rho\mu\nu\sigma} p_\nu p'_\sigma \\
\gamma_5 & : & \emptyset
\end{array}$$

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- Use **Dirac equations** for u and $\bar{u}$:

$$\bar{u}(p')(\not{p}' - m) = 0 \text{ and } (\not{p} - m) u(p) = 0$$

Unpolarized elastic scattering at Born order.

Parameterization of the matrix element: spin-1/2 case (1/2).

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- Need at most 3 **dimensionless** coefficients a, b and c:

$$\langle N, p' \left| J_\mu^{\mathrm{e.m.}}(0) \right| N, p \rangle = \bar{u}(p') \left(a \frac{q_\mu}{M} + b \gamma_\mu + c \frac{\sigma_{\mu\nu} q^\nu}{M} \right) u(p)$$

Unpolarized elastic scattering at Born order.

Parameterization of the matrix element: spin-1/2 case (2/2).

- Need at most 3 **dimensionless** coefficients a, b and c:

$$\langle N, p' \left| J_\mu^{\mathrm{e.m.}}(0) \right| N, p \rangle = \bar{u}(p') \left(a \frac{q_\mu}{M} + b\gamma_\mu + c \frac{\sigma_{\mu\nu} q^\nu}{M} \right) u(p)$$

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- Enforce **current conservation** $q^\mu J^{\mathrm{e.m.}}_\mu = 0$ with $q^2 < 0$:

$$\bar{u}(p') \left(a \frac{q^2}{M} + b\not{q} + c\frac{\sigma_{\mu\nu} q^\nu q^\mu}{M} \right) u(p) = 0$$

where $\bar{u}(p')(\not{p}' - \not{p})u(p) = 0$ (Dirac equation) and $\sigma_{\mu\nu} q^\nu q^\mu = 0$ (symmetry).

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- **Hermiticity**: b is real and c is purely imaginary.
- b and c **depend on q^2 only** ($-q^2 = 2p \cdot q$ for elastic scattering).

$$\langle N \left| J_\mu^{\text{em}}(0) \right| N \rangle = \bar{u}(p') \left(F_1(Q^2)\gamma_\mu + F_2(Q^2) \frac{i}{2M} \sigma_{\mu\nu} q^\nu \right) u(p)$$

Pauli-Dirac parameterization

$$\langle N | J^{\rm em}_\mu(0) | N \rangle = \bar{u}(p') \left(F_1(Q^2)\gamma_\mu + F_2(Q^2)\frac{i}{2M}\sigma_{\mu\nu}q^\nu \right) u(p)$$

Sachs parameterization

$$\langle N | J^{\rm em}_\mu(0) | N \rangle = \bar{u}(p') \left(\frac{G_E(Q^2) - \tau GM(Q^2)}{1-\tau}\frac{P_\mu}{M} + G_M(Q^2)\frac{i}{2M}\sigma_{\mu\nu}q^\nu \right) u(p)$$

with $\tau = Q^2/(4M^2)$ and:

$$G_E(Q^2) = F_1(Q^2) + \frac{Q^2}{4M^2}F_2(Q^2),$$

$$G_M(Q^2) = F_1(Q^2) + F_2(Q^2).$$

Mott cross section

Scattering of a relativistic electron on a point-like spinless particle:

$$\left.\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{\mathrm{Mott}} = \frac{Q^2\alpha^2\cos^2\frac{\theta}{2}}{4E^2\sin^4\frac{\theta}{2}}\frac{E'}{E}$$

Rosenbluth cross section

Scattering of a relativistic electron on a spin-1/2 composite target:

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \left.\frac{d\sigma}{d\Omega}\right)_{\mathrm{Mott}}\left(\frac{G_E^2(Q^2)+\tau G_M^2(Q^2)}{1+\tau}+2\tau G_M^2(Q^2)\tan^2\frac{\theta}{2}\right)$$

with: $\tau \equiv Q^2/(4M^2)$

Rosenbluth cross section

Scattering of a relativistic electron on a spin-1/2 composite target:

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{d\sigma}{d\Omega}\Big)_{\mathrm{Mott}} \left(\frac{G_E^2(Q^2) + \tau G_M^2(Q^2)}{1+\tau} + 2\tau G_M^2(Q^2) \tan^2\frac{\theta}{2} \right)$$

with: $\tau \equiv Q^2/(4M^2)$

Exercise 0.2

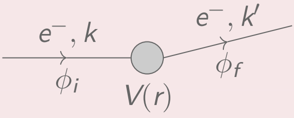
Establish the relation between the energies E and E' of the incoming and outgoing electrons and the scattering angle θ. Comment on the number of independent kinematic variables.

$$E' = \frac{E}{1 + \frac{2E}{M}\sin^2\frac{\theta}{2}}$$

Interpretation of form factors.

Nonrelativistic scattering off a spherically symmetric potential.

Nonrelativistic scattering (scalar particle)

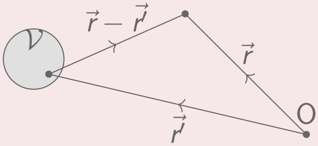


$$\frac{\mathrm{d}\sigma}{\mathrm{d}k'\mathrm{d}\Omega'} \propto |\langle f|V|i\rangle|^2$$

$$\langle f|V|i\rangle = \int \mathrm{d}^3\vec{r} e^{-i\vec{k}' \cdot \vec{r}} V(r) e^{i\vec{k} \cdot \vec{r}}$$

$$\vec{q} = \vec{k} - \vec{k}'$$

Spherically symmetric charge distribution



$$V(r) = \frac{Ze^2}{4\pi} \int_{\mathcal{V}} \mathrm{d}^3\vec{r}' \frac{\rho(r')}{|\vec{r} - \vec{r}'|}$$

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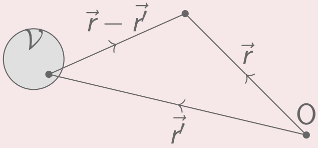
Spherically symmetric charge distribution



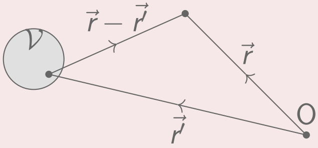
$$V(r) = \frac{Ze^2}{4\pi} \int_{\mathcal{V}} d^3\vec{r}' \frac{\rho(r')}{|\vec{r} - \vec{r}'|}$$

$$\langle f | V | i \rangle = \int d^3\vec{r} e^{-i\vec{q} \cdot \vec{r}} V(r)$$

- Compute in spherical coordinates:

$$\langle f | V | i \rangle = Ze^2 \int_{\mathcal{V}} d^3\vec{r}' e^{i\vec{q} \cdot \vec{r}'} \rho(r') \int_0^{+\infty} dR\, R \frac{\sin qR}{qR}$$

Spherically symmetric charge distribution



$$V(r) = \frac{Ze^2}{4\pi} \int_{\mathcal{V}} \mathrm{d}^3\vec{r}' \frac{\rho(r')}{|\vec{r} - \vec{r}'|}$$

$$\langle f | V | i \rangle = \int \mathrm{d}^3\vec{r} e^{-i\vec{q} \cdot \vec{r}} V(r)$$

- Compute in spherical coordinates: Diverge!

$$\langle f | V | i \rangle = Ze^2 \int_{\mathcal{V}} \mathrm{d}^3\vec{r}'\, e^{i\vec{q} \cdot \vec{r}'} \rho(r') \int_0^{+\infty} \mathrm{d}R\, R \frac{\sin qR}{qR}$$

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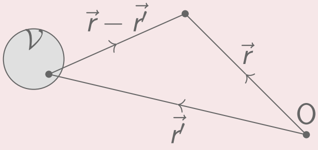
Spherically symmetric charge distribution



$$V(r) = \frac{Ze^2}{4\pi} \int_{\mathcal{V}} \mathrm{d}^3\vec{r}' \frac{\rho(r') e^{-\frac{|\vec{r}-\vec{r}'|}{a}}}{|\vec{r} - \vec{r}'|}$$

$$\langle f | V | i \rangle = \int \mathrm{d}^3\vec{r} e^{-i\vec{q} \cdot \vec{r}} V(r)$$

- Compute in spherical coordinates:

$$\langle f | V | i \rangle = Ze^2 \int_{\mathcal{V}} \mathrm{d}^3\vec{r}' \, e^{i\vec{q} \cdot \vec{r}'} \rho(r') \int_0^{+\infty} \mathrm{d}R \, R \frac{\sin qR}{qR}$$

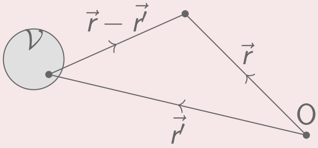
- Regularize: Yukawa screening ($a \simeq 10^{-10}$ $m \simeq 0.5$ keV^{-1})

$$\langle f | V | i \rangle = \frac{Ze^2}{q^2 + \frac{1}{a^2}} F(Q^2) \quad \text{with } F(Q^2) = \int_{\mathcal{V}} \mathrm{d}^3\vec{r} \rho(\vec{r}) e^{i\vec{q} \cdot \vec{r}}$$

Interpretation of form factors.

Nonrelativistic scattering off a spherically symmetric potential.

Spherically symmetric charge distribution



$$\begin{aligned} V(r) &= \frac{Ze^2}{4\pi} \int_{\mathcal{V}} \mathrm{d}^3\vec{r}' \frac{\rho(r')}{|\vec{r} - \vec{r}'|} \\ \langle f | V | i \rangle &= \int \mathrm{d}^3\vec{r} e^{-i\vec{q} \cdot \vec{r}} V(r) \end{aligned}$$

- Compute in spherical coordinates:

$$\langle f | V | i \rangle = Ze^2 \int_{\mathcal{V}} \mathrm{d}^3\vec{r}'\, e^{i\vec{q} \cdot \vec{r}'} \rho(r') \int_0^{+\infty} \mathrm{d}R\, R \frac{\sin qR}{qR}$$

- Regularize: Yukawa screening ($a \simeq 10^{-10}\ m \simeq 0.5\ \mathrm{keV}^{-1}$)

$$\begin{aligned} \langle f | V | i \rangle &= \frac{Ze^2}{q^2 + \frac{1}{a^2}} F(Q^2) \quad \text{with } F(Q^2) = \int_{\mathcal{V}} \mathrm{d}^3\vec{r} \rho(\vec{r}) e^{i\vec{q} \cdot \vec{r}} \\ &\simeq \frac{Ze^2}{q^2} F(Q^2) \quad \text{for } Q \simeq 1.\ \mathrm{GeV} \end{aligned}$$

Rutherford cross section

$$\frac{d\sigma}{d\Omega} = \left(\frac{Z\alpha}{2E}\right)^2 \frac{1}{\sin^2\frac{\theta}{4}} |F(Q^2)|^2$$

where F is the **3D Fourier transform of the target charge distribution**.

Exercise 0.3

Consider $\rho(r) = Ce^{-mr}$ where $m > 0$ and C is such that the total charge is normalized to 1.
Show that $F(Q^2) = 1/(1+Q^2/m^2)^2$ (**dipole** parameterization).

- Take **proton state** with momentum k: $|p, k\rangle$.
- Consider **charge operator**: $\mathbb{Q}\,|p, k\rangle = +\,|p, k\rangle$

$$\mathbb{Q} = \int \mathrm{d}^3\vec{r}\, J_0^{\mathrm{e.m.}}(\vec{r})$$

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- Then $\langle p, k' \,|\mathbb{Q}|\, p, k\rangle = \langle k' \,|k\rangle = 2E_{\vec{k}}(2\pi)^3\delta^{(3)}(\vec{k'} - \vec{k})$ and

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- Then $\langle p, k' |\mathbb{Q}| p, k\rangle = \langle k' | k\rangle = 2E_{\vec{k}}(2\pi)^3\delta^{(3)}(\vec{k'} - \vec{k})$ and

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$$= (2\pi)^3\delta^{(3)}(\vec{k'} - \vec{k})\bar{u}(k)\gamma_0 F_1(0) u(k)$$

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- The form factor F_1 at zero momentum transfer is the **electric charge**.

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\end{aligned}$$

- The form factor F_1 at zero momentum transfer is the **electric charge**.
- Similarly, the form factor F_2 is normalized to the **anomalous magnetic moment**.

Breit frame

$$\vec{p}' = -\vec{p}$$

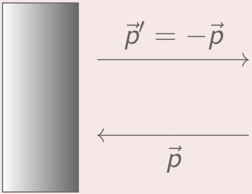
$$\vec{p}$$

Frame in which the **outgoing** nucleon has a 3-momentum opposite to that of the **incoming** nucleon.

Interpretation of form factors.

Nucleon form factors in the Breit frame.

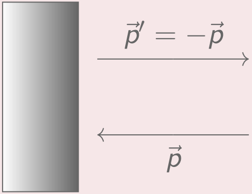
Breit frame



Frame in which the **outgoing** nucleon has a 3-momentum opposite to that of the **incoming** nucleon.

"Brick wall condition"

Breit frame



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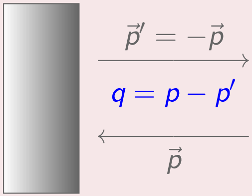

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$$(\text{Gordon id.}) = \bar{u}(p')\left((F_1+F_2)\gamma_0 - F_2\frac{(p+p')_0}{2M}\right)u(p)$$

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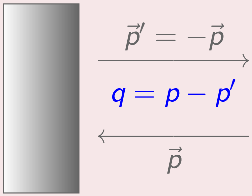

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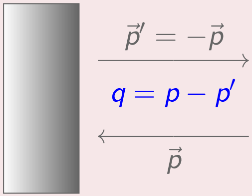
Breit frame



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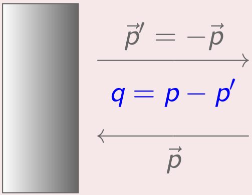
$$= \bar{u}(p')\left((F_1+F_2)\gamma_0 - F_2\frac{(p+p')_0}{2M}\right)u(p)$$

$$(\text{Breit fr.}) = 2M\delta_{hh'}\left[F_1 + F_2\left(1 - \frac{E_p^2}{M^2}\right)\right]$$

Breit frame



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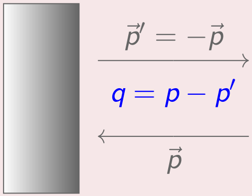
Breit frame



Frame in which the **outgoing** nucleon has a 3-momentum opposite to that of the **incoming** nucleon.

"Brick wall condition"

- Evaluate matrix element of $J^{\mathrm{e.m.}}$ in the Breit frame:

$$
\begin{aligned}
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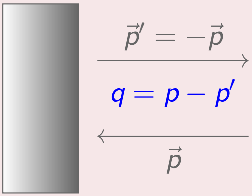
&= \bar{u}(p') \left((F_1 + F_2) \gamma_0 - F_2 \frac{(p + p')_0}{2M} \right) u(p) \\
(q^2 = -4|\vec{p}|^2) \quad &= 2M \delta_{hh'} \left[F_1 + F_2 \left(1 - \frac{E_p^2}{M^2} \right) \right]
\end{aligned}
$$

Breit frame



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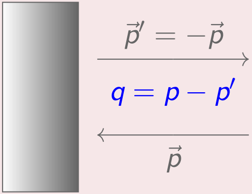
&= \bar{u}(p') \left((F_1 + F_2)\gamma_0 - F_2 \frac{(p + p')_0}{2M} \right) u(p) \\
&= 2M \delta_{hh'} G_E
\end{aligned}
$$

Interpretation of form factors.

Nucleon form factors in the Breit frame.

Breit frame



Frame in which the **outgoing** nucleon has a 3-momentum opposite to that of the **incoming** nucleon.

"Brick wall condition"

Nucleon form factors in the Breit frame

- G_E is the 3D Fourier transform of the **charge density**.
- G_M is the 3D Fourier transform of the **magnetization density**.

- Form factors are **3D Fourier transforms** of distributions in the Breit frame.
- For a **spherically symmetric** charge distribution ρ:

$$
\begin{aligned}
F(Q^2) &= \int_0^{+\infty} dr\,\rho(r) 4\pi r^2 \frac{\sin qr}{qr} \\
&= \int_0^{+\infty} dr\,\rho(r) 4\pi r \frac{1}{q} \left(qr - \frac{q^3 r^3}{6} + \dots \right) \\
&\simeq \int_0^{+\infty} dr\, 4\pi r^2 \rho(r) - \frac{q^2}{6} \int_0^{+\infty} dr\, 4\pi r^2 r^2 \rho(r) + \dots \\
&= 1 - \frac{q^2}{6} \langle r^2 \rangle + \dots
\end{aligned}
$$

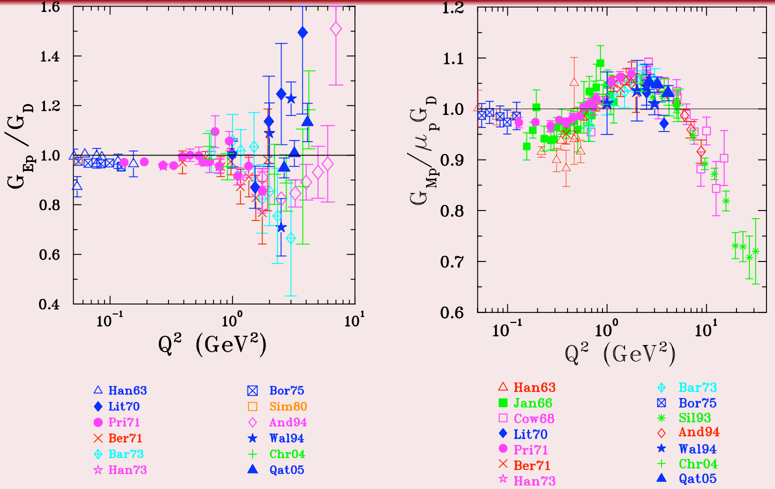
- **Define a charge radius** by:

$$\langle r^2 \rangle \equiv -6 \left. \frac{dF}{dq^2} \right|_{q^2=0}$$

Nucleon charge radius.

Charge radius from elastic scattering measurements.

Measurements



✍ Perdrisat *et al.* (2007)

What have we learnt so far?

- Start with the matrix element of a **conserved current**.
- Enforce **symmetry principles** to parameterize the matrix element with a restricted set of elastic **form factors** (EFFs).
- Relate normalization of EFF to conserved electric charge.
- **Interpret** EFFs in a **particular frame**.
- Define electric **charge radius**.
- Identify a scattering process to measure EFFs.
- Use a **dipole Ansatz** for simple orders of magnitude.

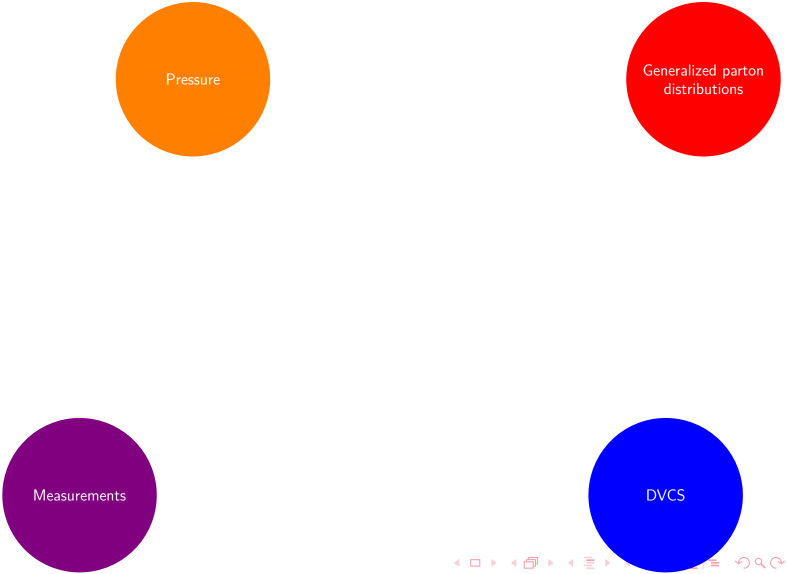
What have we learnt so far?

From electric charge to pressure.

- Start with the matrix element of a **conserved current**.
  Energy-momentum tensor (EMT)
- Enforce **symmetry principles** to parameterize the matrix element with a restricted set of elastic **form factors** (EFFs).
  Gravitational form factors (GFFs)
- Relate normalization of EFF to conserved electric charge.
  Energy and momentum
- **Interpret** EFFs in a **particular frame**.
  Breit frame (not restrictive)
- Define electric **charge radius**.
  Mechanical radius
- Identify a scattering process to measure EFFs.
  Deeply virtual Compton scattering (DVCS) depending on Generalized parton distributions (GPDs)
- Use a **dipole Ansatz** for simple orders of magnitude.

What is the proton internal pressure?

Identifying the concepts.



1. **Energy-momentum tensor**
   Gravitational form factors and pressure distribution.

   Go to Part I.

2. **Generalized parton distributions**
   An indirect way to access gravitational form factors.

   Go to Part II.

3. **Deeply virtual Compton scattering**
   Scattering processes sensitive to generalized parton distributions.

   Go to Part III.

4. **Extraction of pressure distributions**
   From theory to numbers.

   Go to Part IV.

Monday 5 Sep. 2022
8:45 - 9:45

Part I
Energy-momentum tensor

Gravitational form factors and pressure distribution.

Go to outline.

Phase space distribution function.

Microscopic description of an assembly of particles.

- Massive particles (mass m, particle density N).
- Orders of magnitude:

$$\begin{array}{c}\text{de Broglie}\\ \text{wavelength } \lambda\end{array} \ll \begin{array}{c}\text{Average}\\ \text{distance } d_0\end{array} \ll \begin{array}{c}\text{Typical length}\\ \text{scales } L\end{array}$$

e.g. hydrogen in stellar atmosphere at $T \simeq 10^4$ K:

- $N \simeq 10^{16}$ cm^{-3},
- $d_0 = (4\pi/3N)^{-1/3} \simeq 3 \times 10^{-6}$ cm,
- $L \simeq 100$ km,
- $\lambda = h/\sqrt{3mk_B T} \simeq 2 \times 10^{-9}$ cm.

- Approx.: **continuous distribution of classical particles**.

Distribution function $f(\vec{r}, \vec{v}, t)$

$f(\vec{r}, \vec{v}, t)\, \mathrm{d}^3\vec{r}\mathrm{d}^3\vec{v}$ is the average number of particles contained, at time t, in a volume element $\mathrm{d}^3\vec{r}$ about $\vec{r}$ and velocity-space element $\mathrm{d}^3\vec{v}$ about $\vec{v}$.

- **Macroscopic properties** are computed from the distribution function, *e.g.*:
  - Particle density:

$$N(\vec{r}, t) = \int \mathrm{d}^3\vec{v}\, f(\vec{r}, \vec{v}, t)$$

  - Mass density ρ (atomic weight A):

$$\rho(\vec{r}, t) = A m_H N(\vec{r}, t)$$

  - Average velocity $\langle \vec{v} \rangle$:

$$\langle \vec{v} \rangle (\vec{r}, t) = \int \mathrm{d}^3\vec{v}\, \vec{v} f(\vec{r}, \vec{v}, t)$$

- f is a **1-particle distribution function**: the probability of finding a particle at a given point in phase space us independent of the coordinates of all other particles.
- By construction $f(\vec{r}, \vec{v}, t)$ is **positive**.

- Must modify definition of phase space distribution $f(\vec{r}, \vec{v}, t)$ to satisfy **Heisenberg uncertainty principle**.

Wigner quasiprobability distribution.

Including quantum effects.

- Must modify definition of phase space distribution $f(\vec{r}, \vec{v}, t)$ to satisfy **Heisenberg uncertainty principle**.
- Change **kinetic momentum** $\vec{p} = m\vec{v}$ to **canonical momentum** $\vec{p} = \partial\mathcal{L}/\partial\vec{v}$.

Wigner distribution $\mathcal{W}$ (pure state)

Let ψ be the **wavefunction** of the considered system. The **Wigner distribution** $\mathcal{W}(\vec{r}, \vec{p})$ is:

$$\mathcal{W}(\vec{r}, \vec{p}, t) = \int \frac{\mathrm{d}^3\vec{s}}{(2\pi)^3} \psi^* \left(\vec{r} - \frac{1}{2}\vec{s}, t\right) \psi \left(\vec{r} + \frac{1}{2}\vec{s}, t\right) e^{i\vec{p} \cdot \vec{s}}$$

✍ Wigner (1932)

- Must modify definition of phase space distribution $f(\vec{r}, \vec{v}, t)$ to satisfy **Heisenberg uncertainty principle**.
- Change **kinetic momentum** $\vec{p} = m\vec{v}$ to **canonical momentum** $\vec{p} = \partial\mathcal{L}/\partial\vec{v}$.

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✍ Wigner (1932)

- By construction $\mathcal{W}(\vec{r}, \vec{p}, t)$ is **real** but **not necessarily positive**.

Wigner quasiprobability distribution.

Properties (pure state).

- Recover $\vec{r}$ and $\vec{p}$ **probability densities**:

$$\int \mathrm{d}^3\vec{p}\,\mathcal{W}(\vec{r},\vec{p}) = \int \frac{\mathrm{d}^3\,\vec{s}}{(2\pi)^3}\psi^*\left(\vec{r}-\frac{\vec{s}}{2}\right)\psi\left(\vec{r}+\frac{\vec{s}}{2}\right)\int \mathrm{d}^3\vec{p}e^{i\vec{p}\,\cdot\,\vec{s}}$$

Wigner quasiprobability distribution.

Properties (pure state).

- Recover $\vec{r}$ and $\vec{p}$ **probability densities**:

$$\int \mathrm{d}^3\vec{p}\,\mathcal{W}(\vec{r},\vec{p}) = \int \frac{\mathrm{d}^3\vec{s}}{(2\pi)^3}\psi^*\left(\vec{r}-\frac{\vec{s}}{2}\right)\psi\left(\vec{r}+\frac{\vec{s}}{2}\right)\int \mathrm{d}^3\vec{p}e^{i\vec{p}\cdot\vec{s}}$$

$$= \int \mathrm{d}^3\vec{s}\psi^*\left(\vec{r}-\frac{\vec{s}}{2}\right)\psi\left(\vec{r}+\frac{\vec{s}}{2}\right)\delta^{(3)}(\vec{s})$$

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$$= \psi^*(\vec{r})\psi(\vec{r})$$

- Recover $\vec{r}$ and $\vec{p}$ **probability densities**:

$$\int \mathrm{d}^3\vec{p}\,\mathcal{W}(\vec{r},\vec{p}) = |\psi(\vec{r})|^2$$

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- For an observable A associated to a function $a(\vec{r},\vec{p})$ of **phase-space coordinates**:

$$\langle A \rangle = \int \mathrm{d}^3\vec{r}\mathrm{d}^3\vec{p}\,a(\vec{r},\vec{p})\mathcal{W}(\vec{r},\vec{p})$$

✍ Moyal (1949)

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✍ Moyal (1949)

- **Quantum mechanical generalization** of distribution function $f(\vec{r},\vec{p})$.

Wigner quasiprobability distribution.

Properties (pure state).

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✍ Moyal (1949)

- **Quantum mechanical generalization** of distribution function $f(\vec{r},\vec{p})$.
- Need to consider **mixed states** *e.g.* to take spin into account.

- Consider a system $|\psi\rangle$ which is in state $|k\rangle$ with probability p_k $(1 \leq k \leq K$ and $\sum_1^K p_k = 1)$.
- Choose a complete set of (orthonormal) states $|u_n\rangle$:

$$|k\rangle = \sum_n c_n^{(k)} |u_n\rangle \quad \text{for } 1 \leq k \leq n$$

- Compute average value of observable A in state $|k\rangle$:

$$\langle k |A| k\rangle = \sum_{n,m} c_n^{(k)*} c_m^{(k)} A_{nm} \quad \text{with } A_{nm} = \langle u_n |A| u_m\rangle$$

- **Define** operator ρ by matrix element:

$$\rho_{nm} = \langle u_n |\rho| u_m\rangle = \sum_{k=1}^{K} p_k \, c_n^{(k)*} c_m^{(k)}$$

- By construction:

$$\langle \psi |A| \psi\rangle = \sum_{n,m} \rho_{nm} A_{nm} = \mathrm{Tr}\ \rho A$$

Density operator ρ

Every state can be represented by an **density operator** ρ with the following properties:

1. ρ is hermitian.
2. $\text{Tr } \rho = 1$.
3. ρ is positive:
$$\langle \psi | \rho | \psi \rangle \geq 0 \quad \text{for all states } \psi$$
4. The state is pure if and only if $\rho^2 = \rho$.

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   The average value of $B = AA^\dagger$ is positive.

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   ρ is a projection operator.

Wigner quasiprobability distribution.

Nonrelativistic quantum mechanical definition (mixed state).

- Reminder: definition for a pure state.

$$\mathcal{W}_{\mathrm{pure}}(\vec{r}, \vec{p}, t) = \int \frac{\mathrm{d}^3\vec{s}}{(2\pi)^3} \psi^*\left(\vec{r} - \frac{1}{2}\vec{s}, t\right) \psi\left(\vec{r} + \frac{1}{2}\vec{s}, t\right) e^{i\vec{p}\cdot\vec{s}}$$

Wigner distribution $\mathcal{W}$ (mixed state)

Let ρ be the **density operator** of the considered system. The **Wigner distribution** $\mathcal{W}(\vec{r}, \vec{p})$ is:

$$\mathcal{W}(\vec{r}, \vec{p}) = \int \frac{\mathrm{d}^3\vec{s}}{(2\pi)^3} \left\langle \vec{r} - \frac{1}{2}\vec{s} \middle| \rho \middle| \vec{r} + \frac{1}{2}\vec{s} \right\rangle e^{i\vec{p}\cdot\vec{s}}$$

- Need extensions to describe:
  - Quark fields.
  - Color gauge invariance.
  - Lorentz invariance.

- (Trial) Wigner distribution operator $\hat{\mathcal{W}}$:

$$\hat{\mathcal{W}}_\Gamma\left((t,\vec{r}),p\right) = \int \mathrm{d}^4 s\, \bar{\psi}\left(\vec{r}-\frac{1}{2}\vec{s}\right)\Gamma\psi\left(\vec{r}+\frac{1}{2}\vec{s}\right)e^{ip\cdot s}$$

where $\Gamma = 1$, γ_μ, $\gamma_\mu\gamma_5$ or γ_5.

- Choose a **constant 4-vector** n^μ and a non-singular gauge (gauge potentials vanish at spacetime infinity).
- Connect quark fields at $r \pm s/2$ with a **Wilson line** $\mathcal{L}$ *via* intermediate points at $n\infty$ to **ensure gauge invariance**.
- Sandwich between nucleon states with relativistic normalization:

$$\mathcal{W}_\Gamma\left((t,\vec{r}),p\right) = \frac{1}{2M}\int\frac{\mathrm{d}^3\vec{q}}{(2\pi)^3}\left\langle N,\frac{\vec{q}}{2}\right|\hat{\mathcal{W}}_\Gamma\left((t,\vec{r}),p\right)\left|N,-\frac{\vec{q}}{2}\right\rangle$$

✍ Ji (2003)

✍ Belitsky *et al.* (2004)

- Relativistic normalization of 1-particle states:

$$\langle N, p \,|\, N, k \rangle = (2\pi)^3 2E_{\vec{p}} \delta^{(3)}(\vec{p} - \vec{k})$$

- Use translation operator $\mathbb{P}$: $\phi(x + a) = e^{+i\mathbb{P} \cdot a} \phi(x) e^{-i\mathbb{P} \cdot a}$

$$\mathcal{W}_\Gamma\Big((t, \vec{r}), p\Big) = \frac{1}{2M} \int \frac{\mathrm{d}^3\vec{q}}{(2\pi)^3} e^{-i\vec{q} \cdot \vec{r}} \left\langle N, \frac{\vec{q}}{2} \middle| \hat{\mathcal{W}}_\Gamma\Big((t, \vec{0}), p\Big) \middle| N, -\frac{\vec{q}}{2} \right\rangle$$

- To get a **non-trivial** phase-space dependence on $\vec{r}$, take initial and final hadrons with **different** center-of-mass momenta.

Exercise I.1

Recover the nonrelativistic quantum mechanical definition.

Nonrelativistic Wigner distribution for quarks in QCD.

$$\mathcal{W}_\Gamma\Big((t,\vec{r}),p\Big) = \frac{1}{2M}\int \frac{\mathrm{d}^3\vec{q}}{(2\pi)^3}\left\langle N,\frac{\vec{q}}{2}\middle|\hat{\mathcal{W}}_\Gamma\Big((t,\vec{r}),p\Big)\middle|N,-\frac{\vec{q}}{2}\right\rangle$$

✍ Ji (2003)

- Is it measurable?

▸ See fully relativistic treatment.

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✍ Ji (2003)

- Is it measurable? Not clear!

▸ See fully relativistic treatment.

Wigner quasiprobability distribution.

Nonrelativistic Wigner distribution for quarks in QCD (field theory).

Nonrelativistic Wigner distribution for quarks in QCD.

$$\mathcal{W}_\Gamma\Big((t,\vec{r}),p\Big) = \frac{1}{2M}\int\frac{\mathrm{d}^3\vec{q}}{(2\pi)^3}\left\langle N,\frac{\vec{q}}{2}\middle|\hat{\mathcal{W}}_\Gamma\Big((t,\vec{r}),p\Big)\middle|N,-\frac{\vec{q}}{2}\right\rangle$$

✍ Ji (2003)

- Is it measurable? Not clear!
- It is familiar?

▸ See fully relativistic treatment.

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✍ Ji (2003)

- Is it measurable? Not clear!
- It is familiar? Try with $\Gamma = \gamma_\mu$.

$$\hat{\mathcal{W}}_{\gamma_\mu}\Big((t,\vec{r}),p\Big) = \int \mathrm{d}^4 s\,\bar{\psi}\left(\vec{r}-\frac{1}{2}\vec{s}\right)\gamma_\mu\psi\left(\vec{r}+\frac{1}{2}\vec{s}\right)e^{ip\cdot s}$$

Nonrelativistic Wigner distribution for quarks in QCD.

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$$\int \frac{\mathrm{d}^4p}{(2\pi)^4}\hat{\mathcal{W}}_{\gamma_\mu}\Big((t,\vec{r}),p\Big) = \bar{\psi}\Big(t,\vec{r}\Big)\gamma_\mu\psi\Big((t,\vec{r})\Big)$$

See fully relativistic treatment.

Nonrelativistic Wigner distribution for quarks in QCD.

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✍ Ji (2003)

- Is it measurable? Not clear!
- It is familiar? Yes!

$$\hat{\mathcal{W}}_{\gamma_\mu}\Big((t,\vec{r}),p\Big) = \int \mathrm{d}^4 s\,\bar{\psi}\left(\vec{r}-\frac{1}{2}\vec{s}\right)\gamma_\mu\psi\left(\vec{r}+\frac{1}{2}\vec{s}\right)e^{ip\,\cdot\,s}$$

$$\int \frac{\mathrm{d}^4 p}{(2\pi)^4}\hat{\mathcal{W}}_{\gamma_\mu}\Big((t,\vec{r}),p\Big) = \bar{\psi}\Big(t,\vec{r}\Big)\gamma_\mu\psi\Big((t,\vec{r})\Big)$$

- Matrix element of the **electromagnetic current**!

▸ See fully relativistic treatment.

- EMT defined from the invariance under space and time translations.
- **Quark** and **gluon** contributions

$$
\begin{aligned}
T_q^{\mu\nu} &= \bar{q}\gamma^\mu \frac{i}{2} \overleftrightarrow{\mathrm{D}}\, q \\
T_g^{\mu\nu} &= -F^{\mu\lambda} F^{\nu}{}_{\lambda} + \frac{1}{4}\eta^{\mu\nu} F^2
\end{aligned}
$$

with $\overleftrightarrow{\mathrm{D}}$ the symmetric covariant derivative and $F^{\mu\nu}$ the field strength tensor.

- $T^{\mu\nu} = \sum_a T_a^{\mu\nu}$ $(a = q, g)$.

Parameterization: massive spin-1/2 target.

Introduction of 5 GFFs.

- Local, gauge-invariant, asymmetric EMT:

$$\begin{aligned}\langle p', s' |\, T_a^{\mu\nu}(0)\, | p, s\rangle &= \bar{u}(p', s')\Bigg\{ \frac{P^\mu P^\nu}{M} A_a(t) + M\eta^{\mu\nu}\bar{C}_a(t) \\ &+ \frac{\Delta^\mu \Delta^\nu - \eta^{\mu\nu}\Delta^2}{M} C_a(t) \\ &+ \frac{P^{\{\mu} i\sigma^{\nu\}\Delta}}{4M}\left[A_a(t) + B_a(t)\right] \\ &+ \frac{P^{[\mu} i\sigma^{\nu]\Delta}}{4M} D_a(t) \Bigg\} u(p, s)\end{aligned}$$

with $P = (p' + p)/2$, $\Delta = p' - p$, $t = \Delta^2$ and polarizations s, s'. Shorthand notations: $a^{\{\mu} b^{\nu\}} = a^\mu b^\nu + a^\nu b^\mu$, $a^{[\mu} b^{\nu]} = a^\mu b^\nu - a^\nu b^\mu$, and $i\sigma^{\mu\Delta} = i\sigma^{\mu\lambda}\Delta_\lambda$

✍ Lorcé *et al.* (2018)

Sum rules.

Conséquences of Poincaré invariance.

- Momentum conservation

$$\sum_{a=q,g} A_a(0) = 1$$

- Spin sum rule

$$\sum_{a=q,g} B_a(0) = 0$$

- Non-conservation of partial EMT

$$\sum_{a=q,g} \bar{C}_a(t) = 0$$

since

$$\langle p', s' | \partial_\mu T_a^{\mu\nu}(0) | p, s \rangle = i \Delta^\nu M \bar{u}(p', s') u(p, s) \, \bar{C}_a(t)$$

- Define distribution of a physical quantity inside a system, by first **localizing the system in both position and momentum** space.
- Breit frame where $P^\mu = (P^0, \vec{0})$ and $\Delta^\mu = (0, \vec{\Delta})$

$$\langle T_a^{\mu\nu} \rangle_{\mathrm{BF}(\vec{r})} = \int \frac{\mathrm{d}^3\Delta}{(2\pi)^3} e^{-i\vec{\Delta}\vec{r}} \left[\frac{\langle p', s | T_a^{\mu\nu}(0) | p, s \rangle}{2P^0} \right]_{\vec{P}=\vec{0}}$$

- Specific role of 3D Fourier transform of GFFs.

- **Matrix element** in the Breit frame ($a = q, g$):

$$\left\langle \frac{\Delta}{2} \left| T_a^{\mu\nu}(0) \right| -\frac{\Delta}{2} \right\rangle = M \left\{ \eta^{\mu 0}\eta^{\nu 0} \left[A_a(t) + \frac{t}{4M^2} B_a(t) \right] + \eta^{\mu\nu} \left[\bar{C}_a(t) - \frac{t}{M^2} C_a(t) \right] + \frac{\Delta^\mu \Delta^\nu}{M^2} C_a(t) \right\}$$

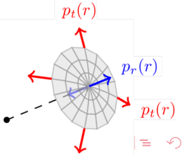
- Anisotropic fluid in **relativistic hydrodynamics**:

$$\Theta^{\mu\nu}(\vec{r}) = \left[\varepsilon(r) + p_t(r)\right] u^\mu u^\nu - p_t(r)\eta^{\mu\nu} + \left[p_r(r) - p_t(r)\right] \chi^\mu \chi^\nu$$

where u^μ and $\chi^\mu = x^\mu / r$.

- Define **isotropic pressure** and **pressure anisotropy**:

$$p(r) = \frac{p_r(r) + 2\, p_t(r)}{3}$$
$$s(r) = p_r(r) - p_t(r)$$



✍ Lorcé et al. (2019)

- Write dictionary between quantum and fluid pictures:

$$\frac{\varepsilon_a(r)}{M} = \int \frac{d^3\vec{\Delta}}{(2\pi)^3} e^{-i\vec{\Delta}\cdot\vec{r}} \left\{ A_a(t) + \bar{C}_a(t) + \frac{t}{4M^2}\left[B_a(t) - 4C_a(t)\right] \right\}$$

$$\frac{p_{r,a}(r)}{M} = \int \frac{d^3\vec{\Delta}}{(2\pi)^3} e^{-i\vec{\Delta}\cdot\vec{r}} \left\{ -\bar{C}_a(t) - \frac{4}{r^2}\frac{t^{-1/2}}{M^2}\frac{d}{dt}\left(t^{3/2}\, C_a(t)\right) \right\}$$

$$\frac{p_{t,a}(r)}{M} = \int \frac{d^3\vec{\Delta}}{(2\pi)^3} e^{-i\vec{\Delta}\cdot\vec{r}} \left\{ -\bar{C}_a(t) + \frac{4}{r^2}\frac{t^{-1/2}}{M^2}\frac{d}{dt}\left[t\frac{d}{dt}\left(t^{3/2}\, C_a(t)\right)\right] \right\}$$

$$\frac{p_a(r)}{M} = \int \frac{d^3\vec{\Delta}}{(2\pi)^3} e^{-i\vec{\Delta}\cdot\vec{r}} \left\{ -\bar{C}_a(t) + \frac{2}{3}\frac{t}{M^2}\, C_a(t) \right\}$$

$$\frac{s_a(r)}{M} = \int \frac{d^3\vec{\Delta}}{(2\pi)^3} e^{-i\vec{\Delta}\cdot\vec{r}} \left\{ -\frac{4}{r^2}\frac{t^{-1/2}}{M^2}\frac{d^2}{dt^2}\left(t^{5/2}\, C_a(t)\right) \right\}$$

✍ Lorcé *et al.* (2019)

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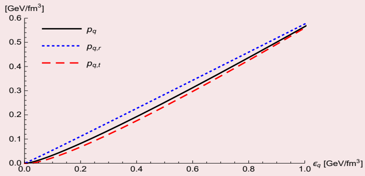
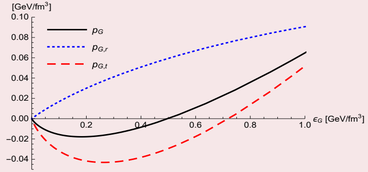
✍ Lorcé *et al.* (2019)

- Evaluate orders of magnitude with naive multiple model:

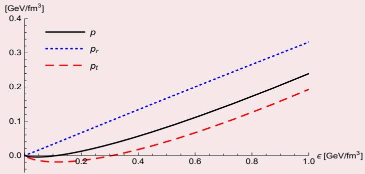
quark



glue

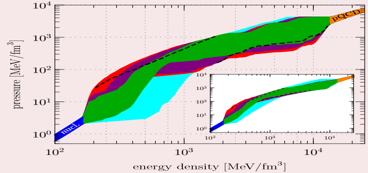


quark + glue



✍ Lorcé *et al.* (2019)

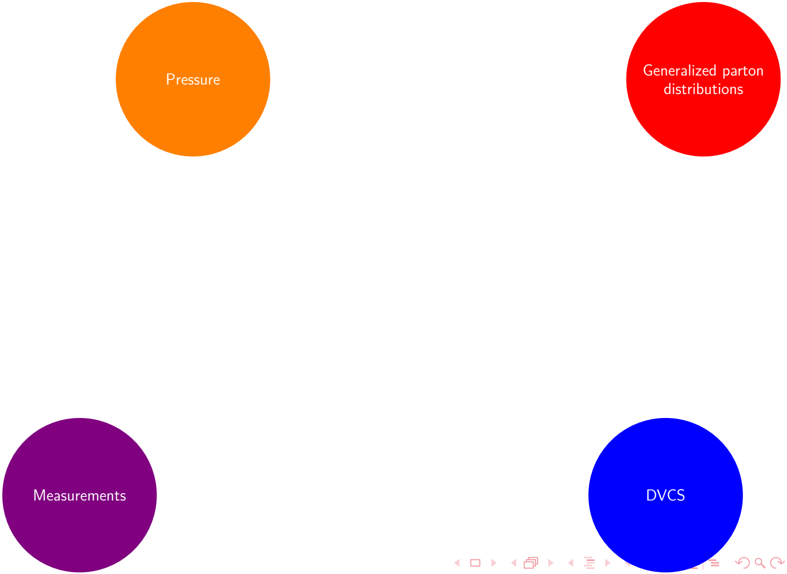
Neutron stars

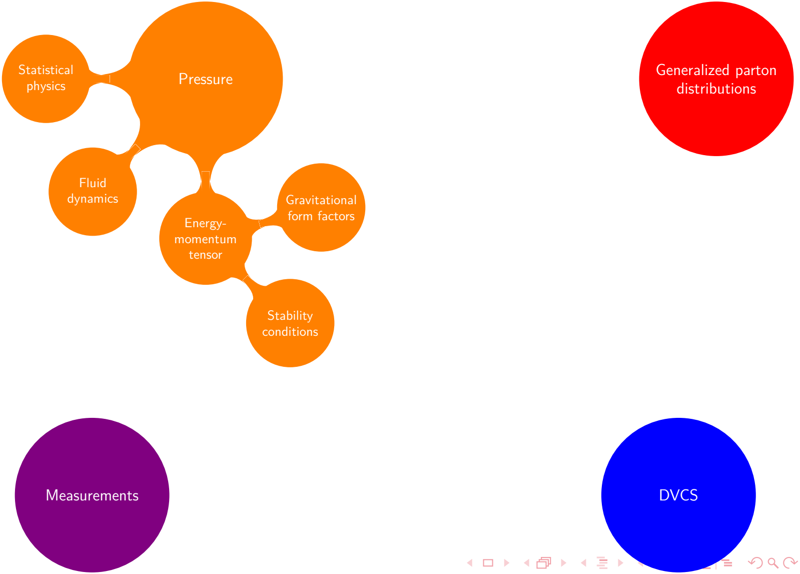


✍ Annala *et al.* (2018)

Summary

What is the proton internal pressure?
Refining the concepts.




Statistical physics
Pressure
Fluid dynamics
Energy-momentum tensor
Gravitational form factors
Stability conditions
Generalized parton distributions
Measurements
DVCS

Abbreviations

Abbreviations used in this part.

DVCS	deeply virtual Compton scattering
EFF	elastic form factor
EMT	energy-momentum tensor
GFF	gravitational form factor
GPD	generalized parton distribution

Monday 5 Sep. 2022
16:00 - 17:00

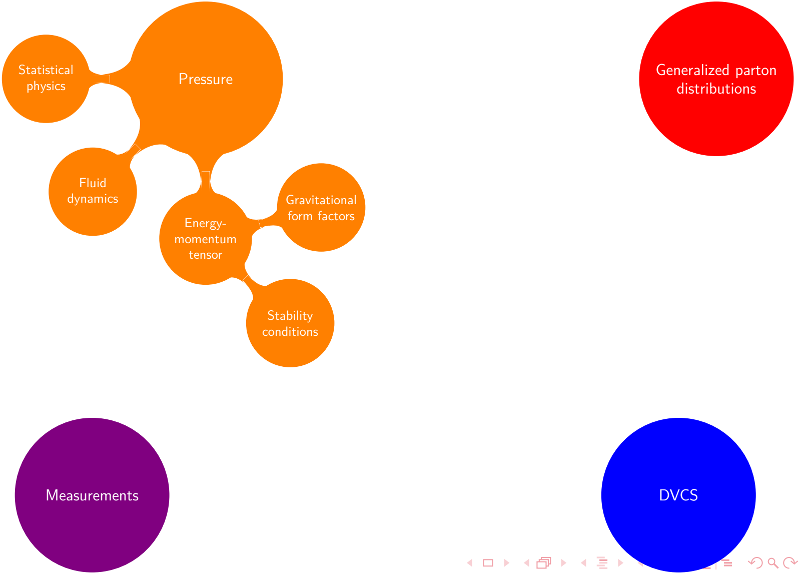
Part II
Generalized parton distributions

An indirect way to access gravitational form factors.

Go to outline.

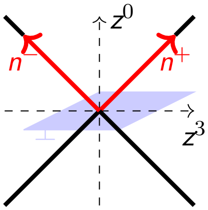
What is the proton internal pressure?

Refining the concepts.



Light-cone coordinates.

Choose privileged axis along which particles have large momentum.



- z axis defined by propagation of **fast moving** particles.
- Write $v^\mu = (v^+, \vec{v}_\perp, v^-)$ for a 4-vector v^μ with:

$$v^+ = \frac{v^0 + v^3}{\sqrt{2}} \text{ and } v^- = \frac{v^0 - v^3}{\sqrt{2}}$$

- Product of two 4-vectors v and w:

$$v \cdot w = v^+ w^- + v^- w^+ - \vec{v}_\perp \cdot \vec{w}_\perp$$

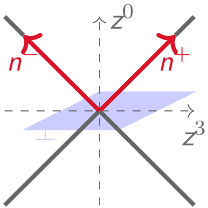
- Take two **light-like** 4-vectors $n_+ = (1, 0, 0, 1)$ and $n_- = (1, 0, 0, -1)$ sucht that:

$$n_+ \cdot n_- = 1 \text{ and } v^\pm = v \cdot n_\mp \text{ for any 4-vector } v^\mu$$

- For a particle moving at the speed of light in the $+z$ direction $(x^3 \simeq x^0)$: $z^- \simeq 0$ and $z^+ \simeq \sqrt{2} x^0$.
- Interpret x^+ as **light-cone time**.

$$H^q_\pi(x,\xi,t) = \frac{1}{2}\int \frac{dz^-}{2\pi}\, e^{ixP^+z^-} \left\langle \pi, P+\frac{\Delta}{2} \middle| \bar{q}\left(-\frac{z}{2}\right)\gamma^+ q\left(\frac{z}{2}\right) \middle| \pi, P-\frac{\Delta}{2} \right\rangle_{\substack{z^+=0\\ z_\perp=0}}$$

with $t=\Delta^2$ and $\xi = -\Delta^+/(2P^+)$.



✍ Müller *et al.* (1994)

✍ Ji (1997)

✍ Radyushkin (1996)

- PDF **forward limit**

$$H^q(x,0,0) = q(x)$$

Spin-0 generalized parton distribution.

Definition and simple properties.

$$H^q_\pi(x,\xi,t) = \frac{1}{2}\int \frac{\mathrm{d}z^-}{2\pi}\, e^{ixP^+z^-} \left\langle \pi, P+\frac{\Delta}{2} \middle| \bar{q}\left(-\frac{z}{2}\right)\gamma^+ q\left(\frac{z}{2}\right) \middle| \pi, P-\frac{\Delta}{2} \right\rangle_{\substack{z^+=0\\ z_\perp=0}}$$

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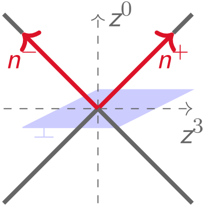


✍ Müller *et al.* (1994)

✍ Ji (1997)

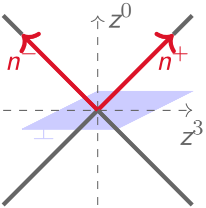
✍ Radyushkin (1996)

- PDF **forward limit**
- Form factor **sum rule**

$$\int_{-1}^{+1} dx\, H^q(x,\xi,t) = F^q_1(t)$$

$$H^q_\pi(x,\xi,t) = \frac{1}{2}\int \frac{dz^-}{2\pi} e^{ixP^+z^-} \left\langle \pi, P+\frac{\Delta}{2} \middle| \bar{q}\left(-\frac{z}{2}\right)\gamma^+ q\left(\frac{z}{2}\right) \middle| \pi, P-\frac{\Delta}{2} \right\rangle_{\substack{z^+=0\\ z_\perp=0}}$$

with $t = \Delta^2$ and $\xi = -\Delta^+/(2P^+)$.



✍ Müller *et al.* (1994)

✍ Ji (1997)

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- PDF **forward limit**
- Form factor **sum rule**
- H^q is an **even function** of ξ from time-reversal invariance.

Spin-0 generalized parton distribution.

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$$H^q_\pi(x, \xi, t) = \frac{1}{2} \int \frac{dz^-}{2\pi} e^{ixP^+z^-} \left\langle \pi, P + \frac{\Delta}{2} \middle| \bar{q}\left(-\frac{z}{2}\right) \gamma^+ q\left(\frac{z}{2}\right) \middle| \pi, P - \frac{\Delta}{2} \right\rangle_{\substack{z^+=0 \\ z_\perp=0}}$$

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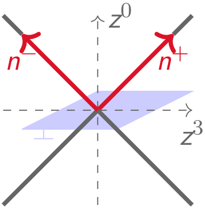


✍ Müller *et al.* (1994)

✍ Ji (1997)

✍ Radyushkin (1996)

- PDF **forward limit**
- Form factor **sum rule**
- H^q is an **even function** of ξ from time-reversal invariance.
- H^q is **real** from hermiticity and time-reversal invariance.

Spin-1/2 generalized parton distributions.

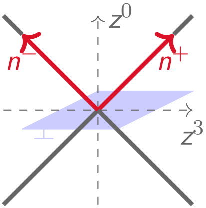
Matrix elements of twist-2 bilocal operators.

$$F^q = \frac{1}{2}\int \frac{dz^-}{2\pi} e^{ixP^+z^-} \langle p' \left| \bar{q}\left(-\frac{z}{2}\right)\gamma^+ q\left(\frac{z}{2}\right)\right| p\rangle_{z^+=0, z_\perp=0}$$

$$= \frac{1}{2P^+}\left[H^q \bar{u}(p')\gamma^+ u(p) + E^q \bar{u}(p')\frac{i\sigma^{+\alpha}\Delta_\alpha}{2M}u(p)\right]$$

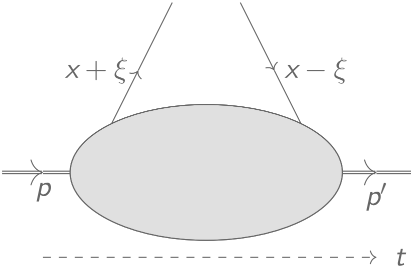
$$\tilde{F}^q = \frac{1}{2}\int \frac{dz^-}{2\pi} e^{ixP^+z^-} \langle p' \left| \bar{q}\left(-\frac{z}{2}\right)\gamma^+\gamma_5 q\left(\frac{z}{2}\right)\right| p\rangle_{z^+=0, z_\perp=0}$$

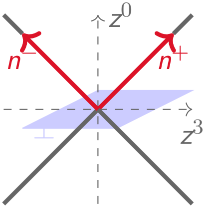
$$= \frac{1}{2P^+}\left[\tilde{H}^q \bar{u}(p')\gamma^+\gamma_5 u(p) + \tilde{E}^q \bar{u}(p')\frac{\gamma^5\Delta^+}{2M}u(p)\right]$$



12 GPDs at twist 2

- Partons with a **light-like** separation.
- **Quarks**, **gluon** and **transversity** GPDs.
- $\mathrm{GPD}^{q,g} = \mathrm{GPD}^{q,g}(x, \xi, t)$.





Interpretation

- $x \in [\xi, 1]$: q emitted + q absorbed.
- $x \in [-\xi, +\xi]$: $\bar{q}$ emitted + q absorbed.
- $x \in [-1, -\xi]$: $\bar{q}$ emitted + $\bar{q}$ absorbed.

Polynomiality

$$\int_{-1}^{+1} dx\, x^n H^q(x, \xi, t) = \text{polynomial in } \xi$$

Polynomiality

Lorentz covariance

See more on polynomiality.

Polynomiality

Lorentz covariance

See more on polynomiality.

Positivity

$$H^q(x, \xi, t) \leq \sqrt{q\left(\frac{x+\xi}{1+\xi}\right) q\left(\frac{x-\xi}{1-\xi}\right)}$$

- **Polynomiality**

Lorentz covariance

See more on polynomiality.

- **Positivity**

Positivity of Hilbert space norm

See more on positivity.

- **Polynomiality**

  Lorentz covariance

  See more on polynomiality.

- **Positivity**

  Positivity of Hilbert space norm

  See more on positivity.

- H^q has support $x \in [-1, +1]$.

Spin-0 generalized parton distribution.

Not so simple properties.

- **Polynomiality**

  Lorentz covariance

  See more on polynomiality.

- **Positivity**

  Positivity of Hilbert space norm

  See more on positivity.

- H^q has support $x \in [-1, +1]$.

  Relativistic quantum mechanics

- **Polynomiality**

  Lorentz covariance

  See more on polynomiality.

- **Positivity**

  Positivity of Hilbert space norm

  See more on positivity.

- H^q has support $x \in [-1, +1]$.

  Relativistic quantum mechanics

- **Soft pion theorem** (pion target)

$$H^q(x, \xi = 1, t = 0) = \frac{1}{2}\phi_\pi^q\left(\frac{1+x}{2}\right)$$

- **Polynomiality**

  Lorentz covariance

  See more on polynomiality.

- **Positivity**

  Positivity of Hilbert space norm

  See more on positivity.

- H^q has support $x \in [-1, +1]$.

  Relativistic quantum mechanics

- **Soft pion theorem** (pion target)

  Dynamical chiral symmetry breaking

- **Polynomiality**

  Lorentz covariance

  ▸ See more on polynomiality.

- **Positivity**

  Positivity of Hilbert space norm

  ▸ See more on positivity.

- H^q has support $x \in [-1, +1]$.

  Relativistic quantum mechanics

- **Soft pion theorem** (pion target)

  Dynamical chiral symmetry breaking

How can we implement *a priori* these theoretical constraints?

- There is no known GPD parameterization **relying only on first principles**.
- In the following, focus on **polynomiality** and **positivity**.

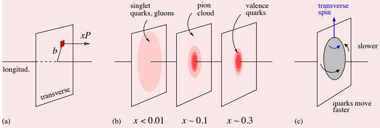
- **Probabilistic interpretation** of Fourier transform of GPD$(x, \xi = 0, t)$ in **transverse plane**.

$$\rho(x, b_\perp, \lambda, \lambda_N) = \frac{1}{2}\left[H(x, 0, b_\perp^2) + \frac{b_\perp^j \epsilon_{ji} S_\perp^i}{M} \frac{\partial E}{\partial b_\perp^2}(x, 0, b_\perp^2) + \lambda\lambda_N \tilde{H}(x, 0, b_\perp^2)\right]$$

- Notations : quark helicity λ, nucleon longitudinal polarization λ_N and nucleon transverse spin $S_\perp$.

✍ Burkardt (2000)

Can we obtain this picture from exclusive measurements?



✍ Weiss (2009)

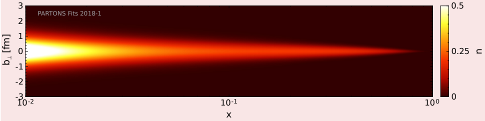
- **Probabilistic interpretation** of Fourier transform of GPD$(x, \xi = 0, t)$ in **transverse plane**.

$$\rho(x, b_\perp, \lambda, \lambda_N) = \frac{1}{2}\left[H(x, 0, b_\perp^2) + \frac{b_\perp^j \epsilon_{ji} S_\perp^i}{M} \frac{\partial E}{\partial b_\perp^2}(x, 0, b_\perp^2) + \lambda\lambda_N \tilde{H}(x, 0, b_\perp^2)\right]$$

- Notations : quark helicity λ, nucleon longitudinal polarization λ_N and nucleon transverse spin $S_\perp$.

✍ Burkardt (2000)

Nucleon tomography in the quark sector

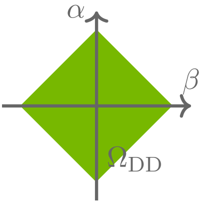


✍ Moutarde *et al.* (2018)

- Define double distributions F^q and G^q as matrix elements of **twist-2 quark operators**:

$$\left\langle P+\frac{\Delta}{2} \middle| \bar{q}(0)\gamma^{\{\mu} i\overleftrightarrow{D}^{\mu_1}\ldots i\overleftrightarrow{D}^{\mu_m\}} q(0) \middle| P-\frac{\Delta}{2} \right\rangle = \sum_{k=0}^{m} \binom{m}{k}$$

$$\left[F^q_{mk}(t) 2P^{\{\mu} - G^q_{mk}(t)\Delta^{\{\mu}\right] P^{\mu_1}\ldots P^{\mu_{m-k}} \left(-\frac{\Delta}{2}\right)^{\mu_{m-k+1}} \ldots \left(-\frac{\Delta}{2}\right)^{\mu_m\}}$$



with

$$F^q_{mk} = \int_{\Omega_{DD}} d\beta d\alpha\, \alpha^k \beta^{m-k} F^q(\beta,\alpha)$$

$$G^q_{mk} = \int_{\Omega_{DD}} d\beta d\alpha\, \alpha^k \beta^{m-k} G^q(\beta,\alpha)$$

✍ Müller *et al.* (1994)
✍ Radyushkin (1999)
✍ Radyushkin (1999)

- Representation of GPD:

$$H^q(x,\xi,t) = \int_{\Omega_{\rm DD}} \mathrm{d}\beta \mathrm{d}\alpha \, \delta(x-\beta-\alpha\xi)\big(F^q(\beta,\alpha,t)+\xi\, G^q(\beta,\alpha,t)\big)$$

- Support property: $x \in [-1,+1]$.
- Discrete symmetries: F^q is α-even and G^q is α-odd.
- **Pobylitsa gauge**: any representation (F^q, G^q) can be recast in one representation with a single DD f^q:

$$H^q(x,\xi,t) = (1-x)\int_{\Omega_{\rm DD}} \mathrm{d}\beta \mathrm{d}\alpha \, f^q(\beta,\alpha,t)\delta(x-\beta-\alpha\xi)$$

✍ Pobylitsa (2003)

✍ Müller (2014)

- Formalism: **Radon transform**.

- Decompose an hadronic state $|H; P, \lambda\rangle$ in a Fock basis:

$$|H; P, \lambda\rangle = \sum_{N,\beta} \int [\mathrm{d}x\mathrm{d}\mathbf{k}_\perp]_N \psi_N^{(\beta,\lambda)}(x_1, \mathbf{k}_{\perp 1}, \ldots, x_N, \mathbf{k}_{\perp N}) \, |\beta, k_1, \ldots, k_N\rangle$$

- Derive an expression for the pion GPD in the DGLAP region $\xi \leq x \leq 1$:

$$H^q(x, \xi, t) \propto \sum_{\beta,j} \int [\mathrm{d}\bar{x}\mathrm{d}\bar{\mathbf{k}}_\perp]_N \delta_{j,q} \delta(x - \bar{x}_j) \left(\psi_N^{(\beta,\lambda)}\right)^* (\hat{x}', \hat{\mathbf{k}}'_\perp) \psi_N^{(\beta,\lambda)}(\tilde{x}, \tilde{\mathbf{k}}_\perp)$$

with $\tilde{x}, \tilde{\mathbf{k}}_\perp$ (resp. $\hat{x}', \hat{\mathbf{k}}'_\perp$) generically denoting incoming (resp. outgoing) parton kinematics.

✍ Diehl *et al.* (2001)

- Similar expression in the ERBL region $-\xi \leq x \leq \xi$, but with overlap of N- and $(N+2)$-body LFWFs.

- Physical picture.
- Positivity relations are fulfilled **by construction**.
- Implementation of **symmetries of N-body problems**.

What is not obvious anymore

What is *not* obvious to see from the wave function representation is however the **continuity of GPDs at** $x = \pm\xi$ and the **polynomiality** condition. In these cases both the DGLAP and the ERBL regions must cooperate to lead to the required properties, and this implies **nontrivial relations between the wave functions** for the different Fock states relevant in the two regions. An *ad hoc* Ansatz for the wave functions would **almost certainly lead** to GPDs that **violate the above requirements**.

✍ Diehl (2003)

- Question: How to access **experimentally** the energy momentum form factors?

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- Consider a **light-like** vector n:

$$\left\langle P+\frac{\Delta}{2}\left|T_q^{\mu\nu}(0)\right|P-\frac{\Delta}{2}\right\rangle n_\mu n_\nu \quad = \left\langle P+\frac{\Delta}{2}\left|\bar{q}\gamma^{\{\mu}i\overleftrightarrow{\mathrm{D}}^{\nu\}}q-\eta^{\mu\nu}\mathcal{L}_{\mathrm{QCD}}\right|P-\frac{\Delta}{2}\right\rangle n_\mu n_\nu$$

Energy momentum form factors.

Projection on the light cone.

- Question: How to access **experimentally** the energy momentum form factors?
- Spin 2 probe: *graviton*?! **Hopeless!**
- Consider a **light-like** vector n:

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- Terms asymmetric w.r.t. $\mu \leftrightarrow \nu$ vanish after contraction with $n_\mu n_\nu$ (notation $\Delta^+ \equiv -2\xi P^+$):

$$\frac{1}{P^{+2}} \left\langle P + \frac{\Delta}{2} \left| \bar{q}(0)\gamma^{(\mu} i \overleftrightarrow{\mathrm{D}}^{\nu)} q(0) \right| P - \frac{\Delta}{2} \right\rangle n_\mu n_\nu = \bar{u}\left(P + \frac{\Delta}{2}\right) \times \left[\frac{A_q(t) + 4\xi^2 C_q(t)}{M} + \left(A_q(t) + B_q(t)\right) i \frac{\sigma^{+\lambda}\Delta_\lambda}{2MP^+} \right] u\left(P - \frac{\Delta}{2}\right)$$

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- Restrict to **symmetric components of EMT**.

- Reminder: Energy momentum tensor

$$\frac{1}{P^{+2}} \left\langle P + \frac{\Delta}{2} \middle| \bar{q}(0)\gamma^{\{+} i\overleftrightarrow{D}^{+\}} q(0) \middle| P - \frac{\Delta}{2} \right\rangle = \bar{u}\left(P + \frac{\Delta}{2}\right)$$

$$\times \left[\frac{A_q(t) + 4\xi^2 C_q(t)}{M} + \left(A_q(t) + B_q(t)\right) i \frac{\sigma^{+\lambda}}{P^+} \frac{\Delta_\lambda}{2M} \right] u\left(P - \frac{\Delta}{2}\right)$$

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- Compute GPDs Mellin moment of order 1:

$$\frac{1}{P^+} \bar{u}(p') \left[\int \mathrm{d}x\, x H^q(x, \xi, t) \gamma^+ + \int \mathrm{d}x\, x E^q(x, \xi, t) \frac{i\sigma^{+\alpha}\Delta_\alpha}{2M} \right] u(p)$$

$$= \int \frac{dz^-}{2\pi} \int \mathrm{d}x\, x e^{ixP^+z^-} \left\langle p' \middle| \bar{q}\left(-\frac{z}{2}\right) \gamma^+ q\left(\frac{z}{2}\right) \middle| p \right\rangle_{z^+=0, z_\perp=0}$$

- Reminder: Energy momentum tensor

$$\frac{1}{P^{+2}}\left\langle P+\frac{\Delta}{2}\right|\bar{q}(0)\gamma^{\{+}i\overleftrightarrow{D}^{+\}}q(0)\left|P-\frac{\Delta}{2}\right\rangle = \bar{u}\left(P+\frac{\Delta}{2}\right)$$
$$\times\left[\frac{A_q(t)+4\xi^2 C_q(t)}{M}+\left(A_q(t)+B_q(t)\right)i\frac{\sigma^{+\lambda}}{P^+}\frac{\Delta_\lambda}{2M}\right]u\left(P-\frac{\Delta}{2}\right)$$

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$$\bar{u}(p')\left[\int \mathrm{d}x\,xH^q(x,\xi,t)\frac{1}{M}+\int \mathrm{d}x\,x(H^q+E^q)(x,\xi,t)\frac{i\sigma^{+\lambda}\Delta_\lambda}{2MP^+}\right]u(p)$$
$$=\int\frac{dz^-}{2\pi}\int \mathrm{d}x\,xe^{ixP^+z^-}\langle p'\left|\bar{q}\left(-\frac{z}{2}\right)\gamma^+ q\left(\frac{z}{2}\right)\right|p\rangle_{z^+=0,z_\perp=0}$$

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$$\times\left[\frac{A_q(t)+4\xi^2 C_q(t)}{M}+\left(A_q(t)+B_q(t)\right)i\frac{\sigma^{+\lambda}}{P^+}\frac{\Delta_\lambda}{2M}\right]u\left(P-\frac{\Delta}{2}\right)$$

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$$=\int\frac{dz^-}{2\pi}2\pi(-i)\delta'(P^+z^-)\langle p'\left|\bar{q}\left(-\frac{z}{2}\right)\gamma^+q\left(\frac{z}{2}\right)\right|p\rangle_{z^+=0,z_\perp=0}$$

- Reminder: Energy momentum tensor

$$\frac{1}{P^{+2}}\left\langle P+\frac{\Delta}{2}\middle|\bar{q}(0)\gamma^{\{+}i\overleftrightarrow{\mathrm{D}}^{+\}}q(0)\middle|P-\frac{\Delta}{2}\right\rangle = \bar{u}\left(P+\frac{\Delta}{2}\right)$$
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$$=\frac{-i}{P^{+2}}\int dz^-\delta'(z^-)\langle p'\left|\bar{q}\left(-\frac{z}{2}\right)\gamma^+ q\left(\frac{z}{2}\right)\right|p\rangle_{z^+=0,z_\perp=0}$$

- Reminder: Energy momentum tensor

$$\frac{1}{P^{+2}}\left\langle P+\frac{\Delta}{2}\right|\bar{q}(0)\gamma^{\{+}i\overleftrightarrow{D}^{+\}}q(0)\left|P-\frac{\Delta}{2}\right\rangle = \bar{u}\left(P+\frac{\Delta}{2}\right)$$

$$\times\left[\frac{A_q(t)+4\xi^2C_q(t)}{M}+\left(A_q(t)+B_q(t)\right)i\frac{\sigma^{+\lambda}}{P^+}\frac{\Delta_\lambda}{2M}\right]u\left(P-\frac{\Delta}{2}\right)$$

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$$\bar{u}(p')\left[\int \mathrm{d}x\, xH^q(x,\xi,t)\frac{1}{M}+\int \mathrm{d}x\, x(H^q+E^q)(x,\xi,t)\frac{i\sigma^{+\lambda}\Delta_\lambda}{2MP^+}\right]u(p)$$

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$$\times\left[\frac{A_q(t)+4\xi^2C_q(t)}{M}+\left(A_q(t)+B_q(t)\right)i\frac{\sigma^{+\lambda}}{P^+}\frac{\Delta_\lambda}{2M}\right]u\left(P-\frac{\Delta}{2}\right)$$

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$$=\frac{+i}{P^{+2}}\frac{\partial}{\partial z^-}\langle p'\left|\bar{q}\left(-\frac{z}{2}\right)\gamma^+q\left(\frac{z}{2}\right)\right|p\rangle_{|z=0}$$

GPDs and EMT (1/2).

Towards a measurement of GFFs.

- Reminder: Energy momentum tensor

$$\frac{1}{P^{+2}}\left\langle P+\frac{\Delta}{2}\middle|\bar{q}(0)\gamma^{\{+}i\overleftrightarrow{D}^{+\}}q(0)\middle|P-\frac{\Delta}{2}\right\rangle = \bar{u}\left(P+\frac{\Delta}{2}\right)$$
$$\times\left[\frac{A_q(t)+4\xi^2C_q(t)}{M}+\left(A_q(t)+B_q(t)\right)i\frac{\sigma^{+\lambda}}{P^+}\frac{\Delta_\lambda}{2M}\right]u\left(P-\frac{\Delta}{2}\right)$$

- Compute GPDs Mellin moment of order 1:

$$\bar{u}(p')\left[\int \mathrm{d}x\, xH^q(x,\xi,t)\frac{1}{M}+\int \mathrm{d}x\, x(H^q+E^q)(x,\xi,t)\frac{i\sigma^{+\lambda}\Delta_\lambda}{2MP^+}\right]u(p)$$
$$=\frac{1}{P^{+2}}\langle p'\left|\bar{q}\,(0)\,\gamma^+ i\overleftrightarrow{D}^+ q\,(0)\right|p\rangle$$

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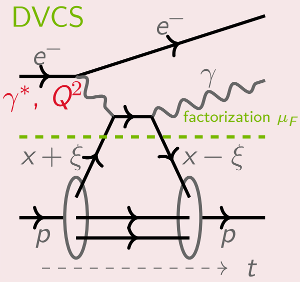
Link between GPDs and GFFs

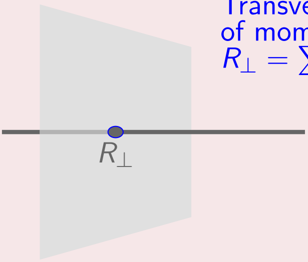
$$\int \mathrm{d}x\, x H^q(x, \xi, t) = A^q(t) + 4\xi^2 C^q(t)$$

$$\int \mathrm{d}x\, x E^q(x, \xi, t) = B^q(t) - 4\xi^2 C^q(t)$$

✍ Ji (1997), ✍ Goeke (2001)

Deeply Virtual Compton Scattering (DVCS)





Transverse center of momentum $R_\perp$

$R_\perp = \sum_i x_i r_{\perp i}$

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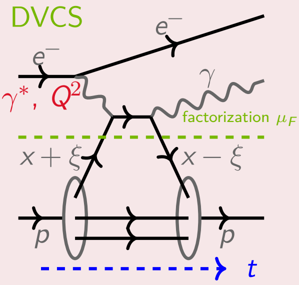

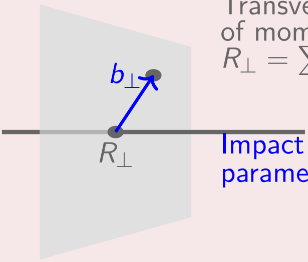

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Impact parameter $b_\perp$

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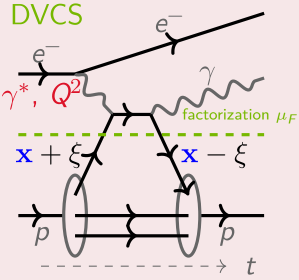

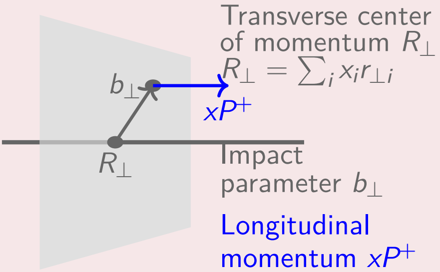

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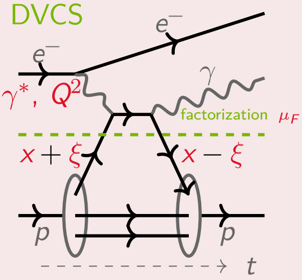

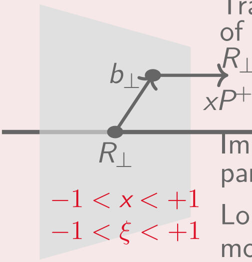

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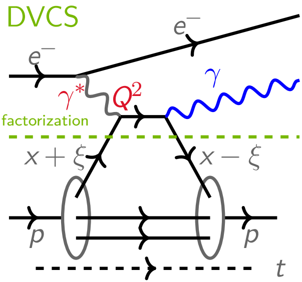
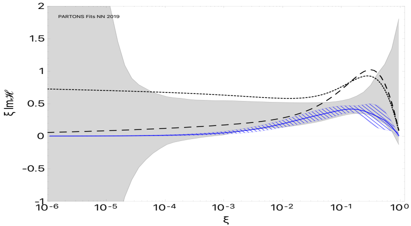
$R_\perp = \sum_i x_i r_{\perp i}$

Impact parameter $b_\perp$

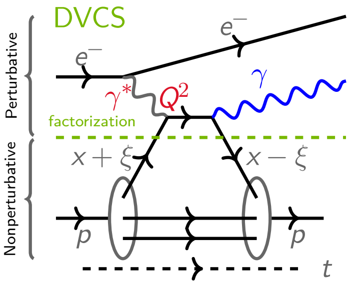
Longitudinal momentum xP^+

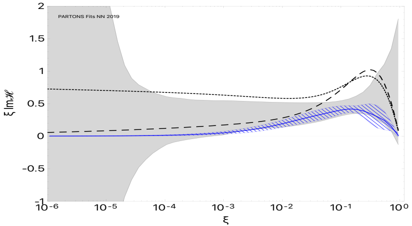
- Mellin moments $\int \mathrm{d}x x^n H(x, \xi, t)$ are polynomials of degree $\leq n+1$ for the GPDs H and E.
- The terms of highest degrees generate the **D-term**.
- The term of highest degree of first Mellin moment $\int \mathrm{d}x x H(x, \xi, t)$of the GPD H is proportional to the GFF C.
- GPD measurements allow an **experimental access** to the EMT.
- The D-term plays a specific role in this strategy.





✍ Moutarde *et al.* (2019)





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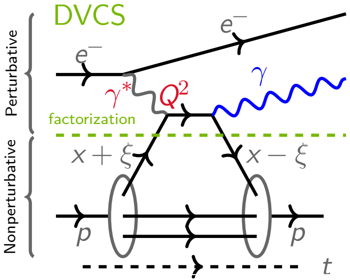

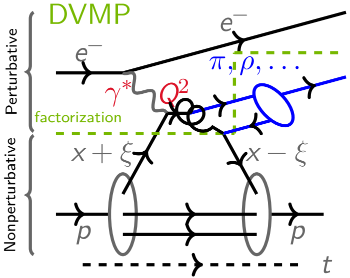

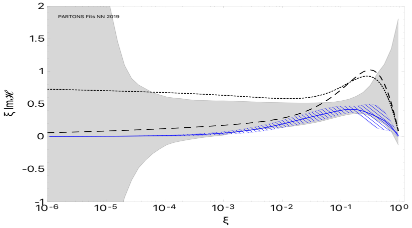

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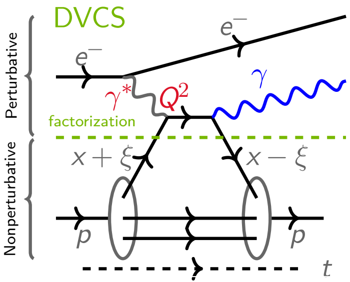

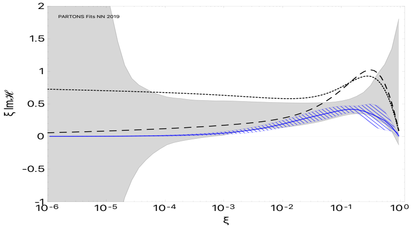

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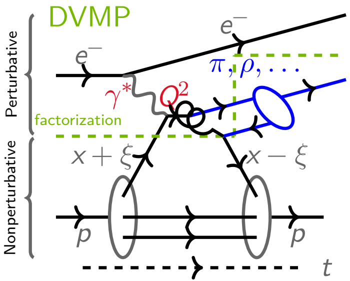
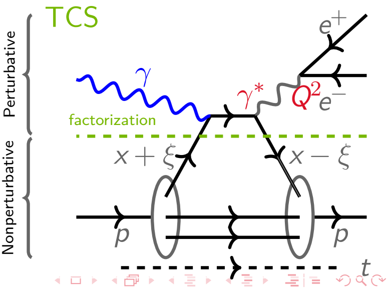
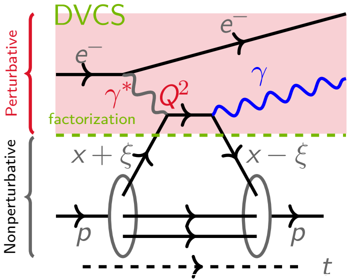
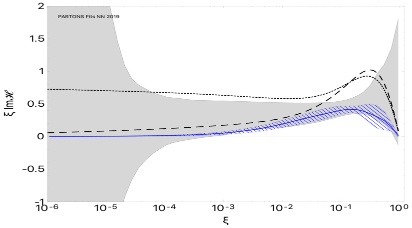
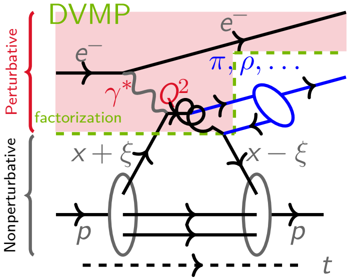


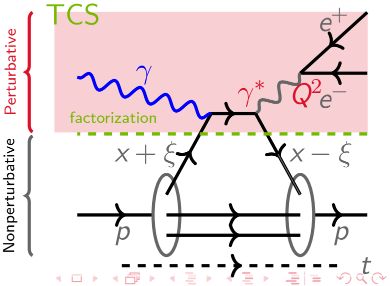


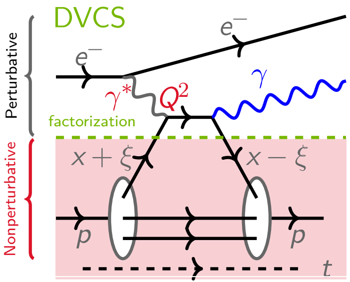


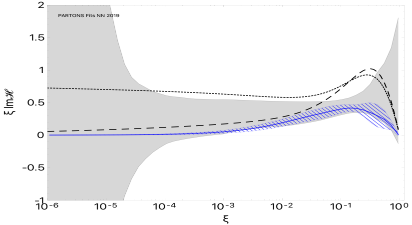


✍ Moutarde *et al.* (2019)

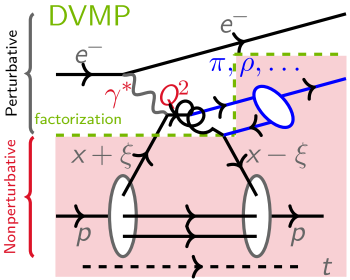


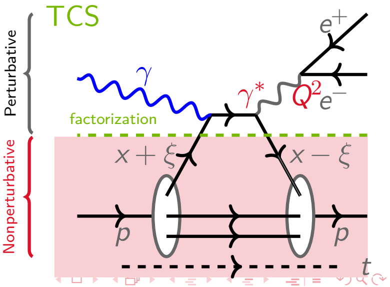


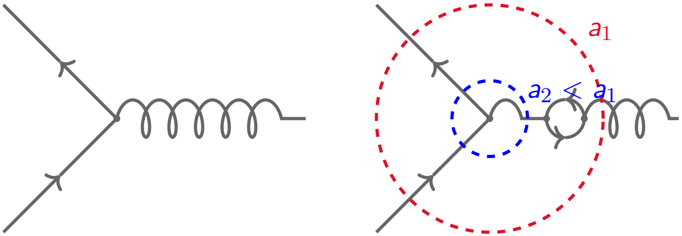




✍ Moutarde *et al.* (2019)







- Each radiative corrections has to be taken into account once and only once.
- Interpretation depends on scale.

- The renormalization of operators defining GPDs requires the introduction of a **factorization scale**.
- This choice defines what is meant by **short** and **large** distance.
- This choice is **arbitrary** and observable quantities do not depend on this scale.
- This remark is materialized through linear differential equations called **evolution equations**.
- The kernel of this equations is computed order by order in **perturbative QCD**.

- Assume CFF $\mathcal{H}$ is perfectly known. Solve inverse problem?

$$\mathcal{H}^q(\xi, Q^2) = \int_{-1}^{1} \frac{\mathrm{d}x}{\xi} T^q\left(\frac{x}{\xi}, \frac{Q^2}{\mu^2}, \alpha_s(\mu^2)\right) H^q(x, \xi, \mu^2)$$

- Question raised about 20 years ago and has remained essentially open. Evolution proposed as a crucial element.

✍ Freund (2000)

- There exist **non-zero** GPDs with **vanishing forward limit** and **vanishing CFF** up to order α_s^2.
- The DVCS deconvolution problem is **ill-posed**.

✍ Bertone *et al.* (2021)

- **Same conclusion holds** for several other hard exclusive processes.
- **Define** and **implement** further criterions in fitting strategies to select one solution among infinitely many.

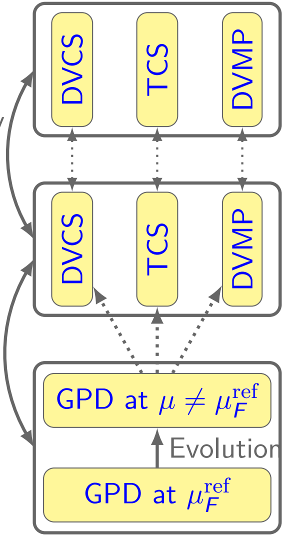
Computing chain design.

EIC perspective considered at the time of design.

Full processes
Experimental data and phenomenology

Small distance
Computation of amplitudes

Large distance
First principles and fundamental parameters



PARtonic **T**omography **O**f **N**ucleon **S**oftware

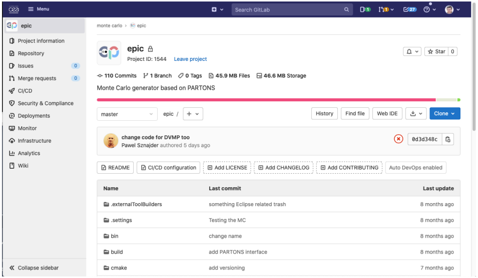
- Perturbative approximations.
- Physical models.
- Fits.
- Numerical methods.
- Accuracy and speed.

✍ Berthou *et al.* (2015)

Generic exclusive event generators EpIC.

Modular structure compatible with the architecture of PARTONS.

- Includes treatment of radiative corrections.
- Can be extended to simulate other exclusive processes.
- **Already used** in the EIC community and run at BNL.
- **Publicly released** simultaneously with PARTONSv3.



✍ Aschenauer *et al.* (2022)

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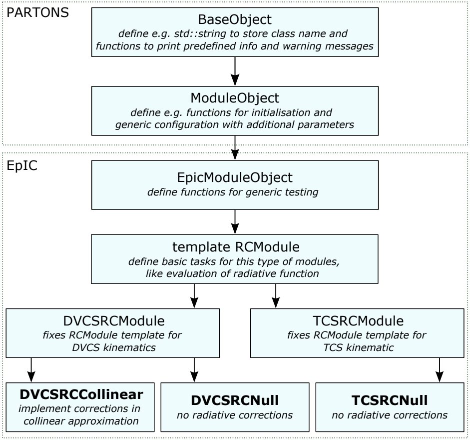

✍ Aschenauer *et al.* (2022)

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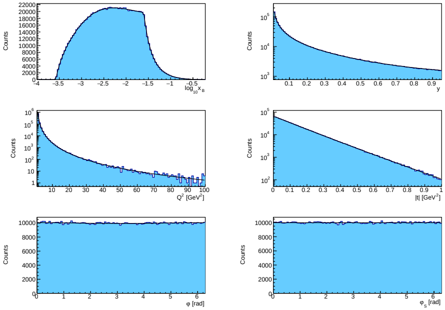
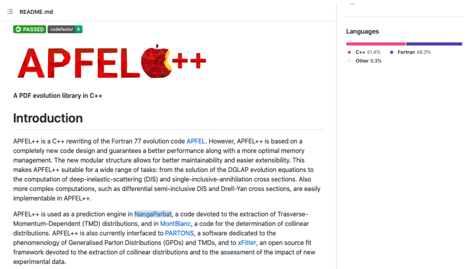


Aschenauer *et al.* (2022)

- Evolution code for PDFs, GPDs and TMDs.
- APFEL++ numerically solves evolution equations in x-space.
- Fully modular.
- Heavy quark threshold crossing.


README.md

PASSED codefactor A

APFEL++

A PDF evolution library in C++

Introduction

APFEL++ is a C++ rewriting of the Fortran 77 evolution code APFEL. However, APFEL++ is based on a completely new code design and guarantees a better performance along with a more optimal memory management. The new modular structure allows for better maintainability and easier extensibility. This makes APFEL++ suitable for a wide range of tasks: from the solution of the DGLAP evolution equations to the computation of deep-inelastic-scattering (DIS) and single-inclusive-annihilation cross sections. Also more complex computations, such as differential semi-inclusive DIS and Drell-Yan cross sections, are easily implementable in APFEL++.

APFEL++ is used as a prediction engine in NangaParbat, a code devoted to the extraction of Trasverse-Momentum-Dependent (TMD) distributions, and in MontBlanc, a code for the determination of collinear distributions. APFEL++ is also currently interfaced to PARTONS, a software dedicated to the phenomenology of Generalised Parton Distributions (GPDs) and TMDs, and to xFitter, an open source fit framework devoted to the extraction of collinear distributions and to the assessment of the impact of new experimental data.

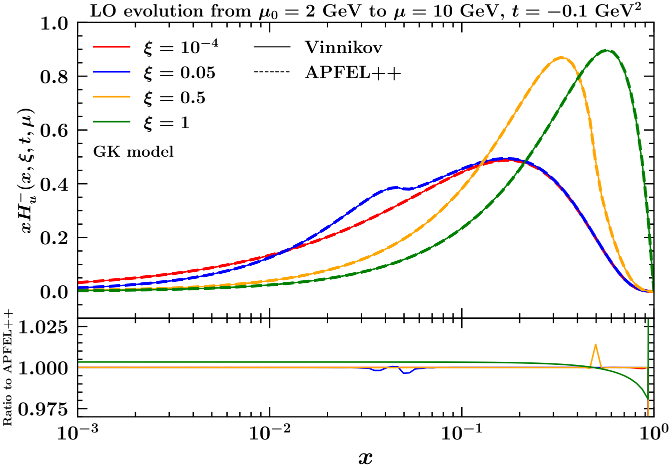
Languages

C++ 51.4% Fortran 48.3% Other 0.3%

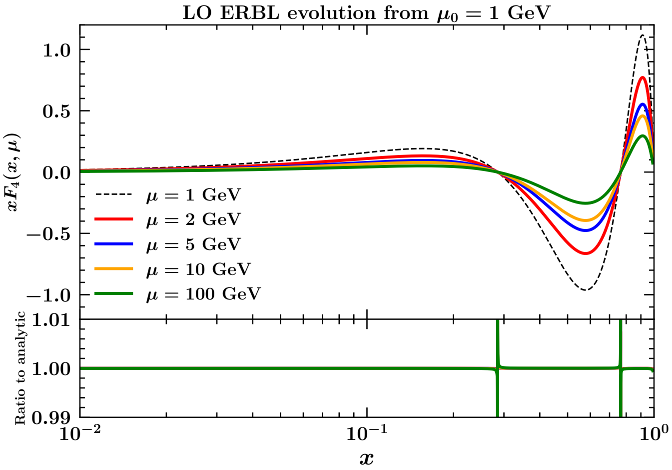
✍ Bertone *et al.* (2022)

GPD evolution with APFEL++.

Connecting different computing codes for hadron structure.



✍ Bertone *et al.* (2022)



✍ Bertone *et al.* (2022)

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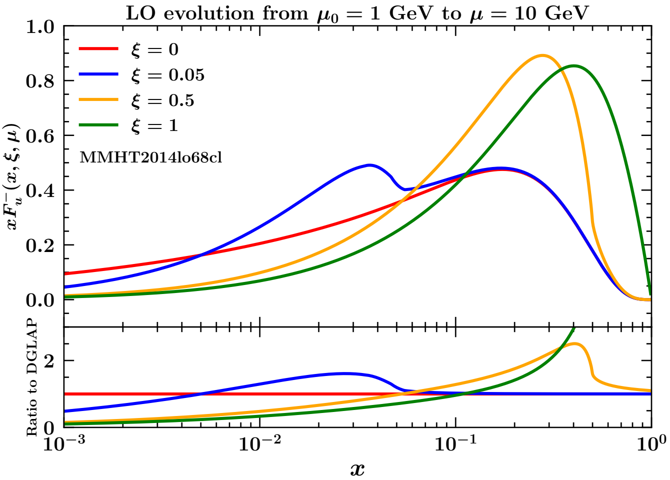
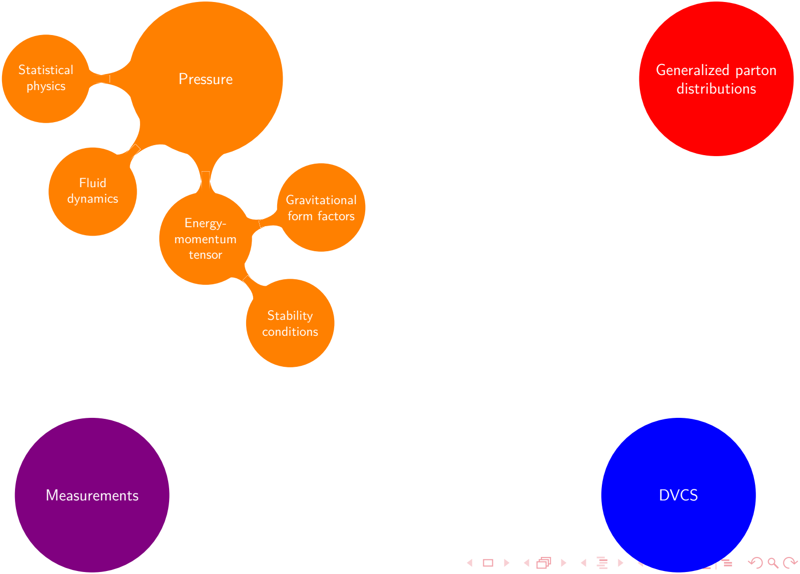


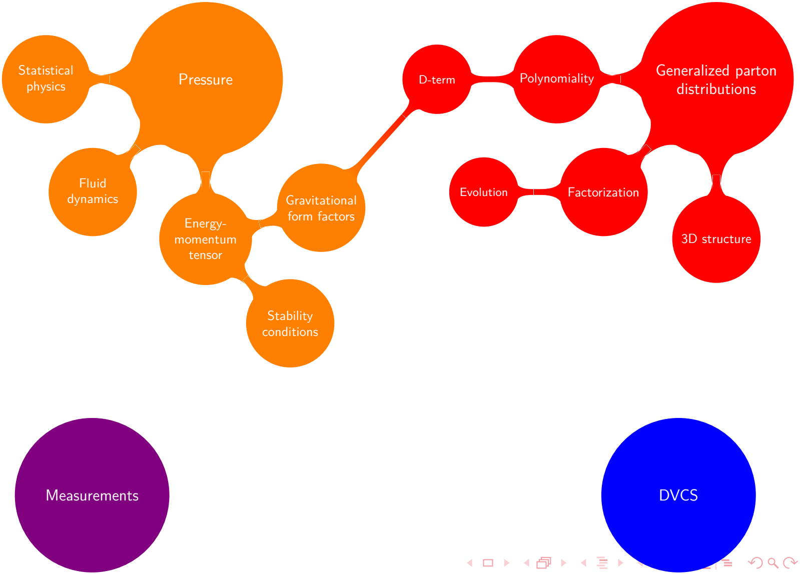
✍ Bertone *et al.* (2022)

Summary

What is the proton internal pressure?

Refining the concepts.




Statistical physics
Pressure
Fluid dynamics
Energy-momentum tensor
Gravitational form factors
Stability conditions
D-term
Polynomiality
Generalized parton distributions
Evolution
Factorization
3D structure
Measurements
DVCS

Abbreviations

Abbreviations used in this part.

APFEL	a PDF evolution library
CFF	Compton form factor
DD	double distribution
DGLAP	Dokshitzer-Gribov-Lipatov-Altarelli-Parisi
DVCS	deeply virtual Compton scattering
DVMP	deeply virtual meson production
EFF	elastic form factor
ERBL	Efremov-Radyushkin-Brodsky-Lepage
GFF	gravitational form factor
GPD	generalized parton distribution
LFWF	light front wave function
TCS	timelike Compton scattering

Tuesday 6 Sep. 2022
10:00 - 11:00

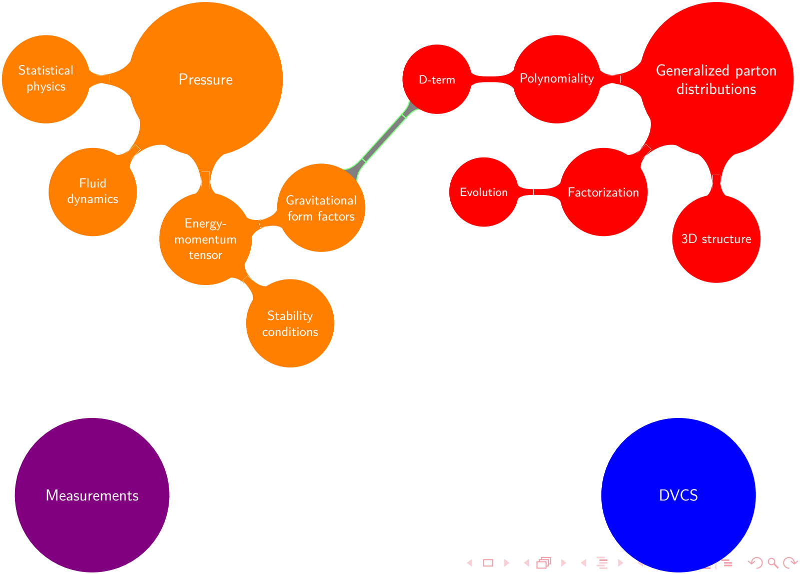
Part III
Deeply virtual Compton scattering

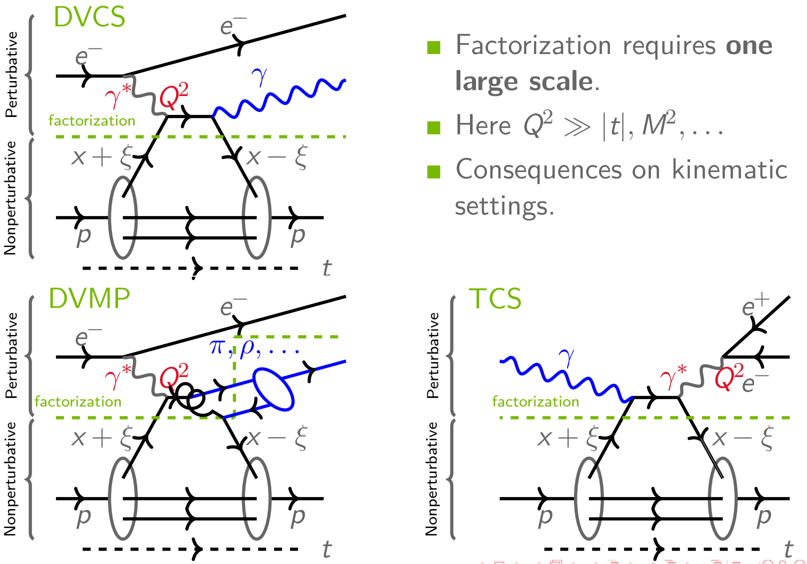
Scattering processes sensitive to generalized parton distributions.

Go to outline.

What is the proton internal pressure?

Refining the concepts.





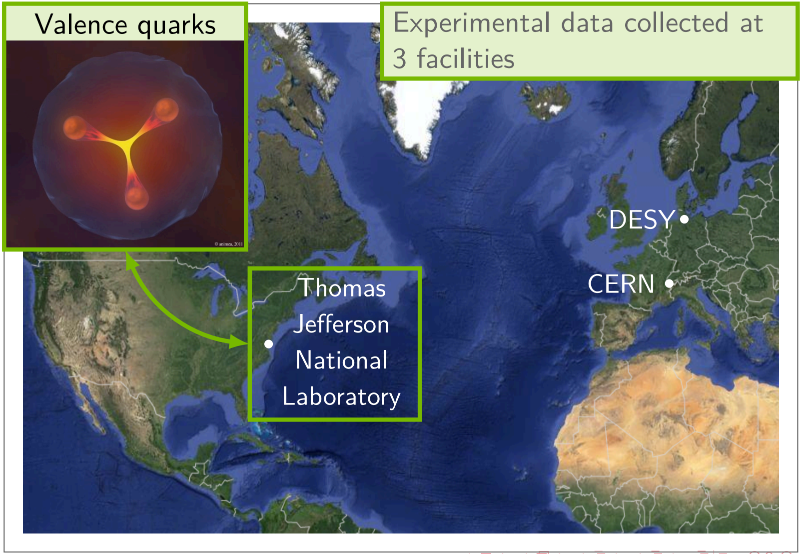
- Factorization requires **one large scale**.
- Here $Q^2 \gg |t|, M^2, \ldots$
- Consequences on kinematic settings.

Need for global fits of world data.
Different facilities will probe different kinematic domains.

Experimental data collected at 3 facilities

Need for global fits of world data.

Different facilities will probe different kinematic domains.



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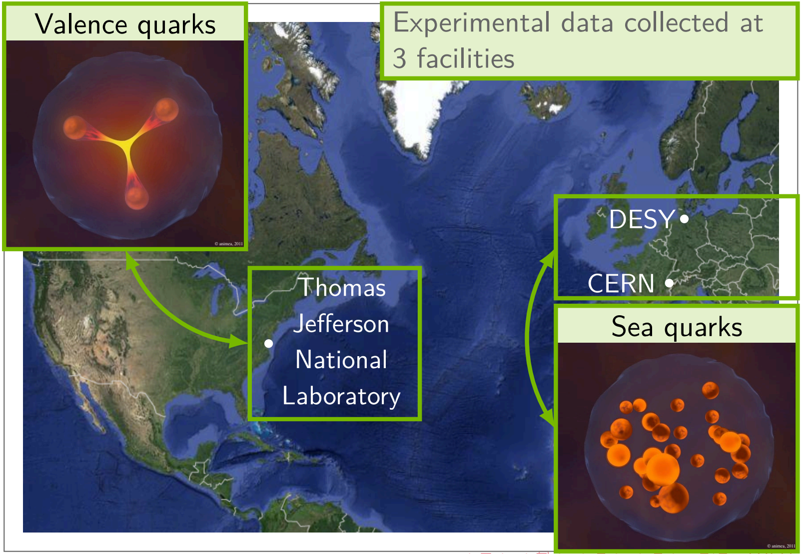
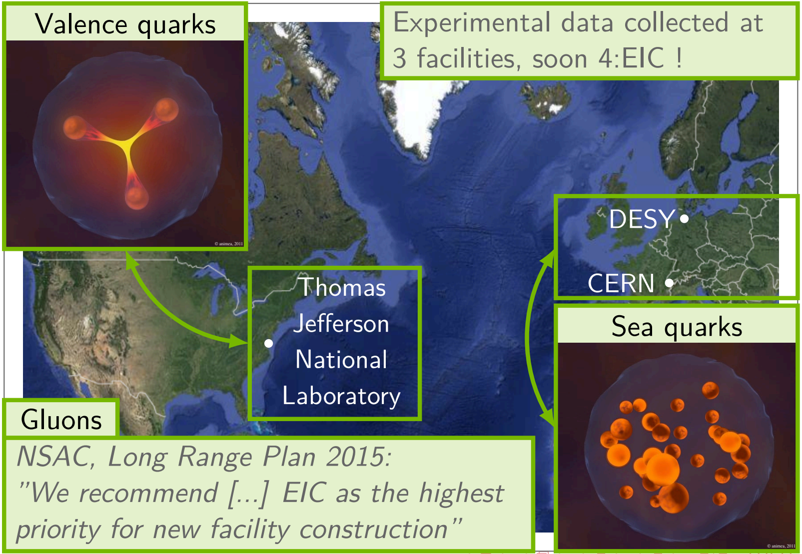


Need for global fits of world data.

Different facilities will probe different kinematic domains.

Experimental data collected at 3 facilities, soon 4:EIC !



NSAC, Long Range Plan 2015: "We recommend [...] EIC as the highest priority for new facility construction"

Almost all existing DVCS data sets.

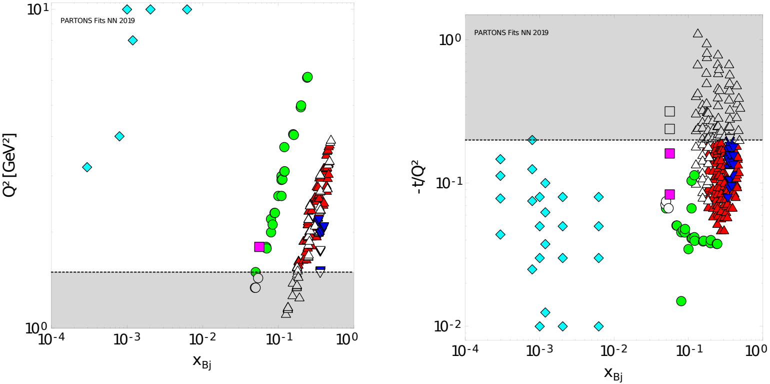
2600+ measurements of 30 observables published during 2001-17.

No.	Collab.	Year	Ref.	Observable		Kinematic dependence	No. of points used / all
1	HERMES	2001	[40]	A^{+}_{LU}		ϕ	10 / 10
2		2006	[41]	$A_{C}^{\cos i\phi}$	$i=1$	t	4 / 4
3		2008	[42]	$A_{C}^{\cos i\phi}$	$i=0,1$	x_{Bj}	18 / 24
				$A_{UT,\mathrm{DVCS}}^{\sin(\phi-\phi_S)\cos i\phi}$	$i=0$		
				$A_{UT,\mathrm{I}}^{\sin(\phi-\phi_S)\cos i\phi}$	$i=0,1$		
				$A_{UT,\mathrm{I}}^{\cos(\phi-\phi_S)\sin i\phi}$	$i=1$		
4		2009	[43]	$A_{LU,\mathrm{I}}^{\sin i\phi}$	$i=1,2$	x_{Bj}	35 / 42
				$A_{LU,\mathrm{DVCS}}^{\sin i\phi}$	$i=1$		
				$A_{C}^{\cos i\phi}$	$i=0,1,2,3$		
5		2010	[44]	$A_{UL}^{+,\sin i\phi}$	$i=1,2,3$	x_{Bj}	18 / 24
				$A_{LL}^{+,\cos i\phi}$	$i=0,1,2$		
6		2011	[45]	$A_{LT,\mathrm{DVCS}}^{\cos(\phi-\phi_S)\cos i\phi}$	$i=0,1$	x_{Bj}	24 / 32
				$A_{LT,\mathrm{DVCS}}^{\sin(\phi-\phi_S)\sin i\phi}$	$i=1$		
				$A_{LT,\mathrm{I}}^{\cos(\phi-\phi_S)\cos i\phi}$	$i=0,1,2$		
				$A_{LT,\mathrm{I}}^{\sin(\phi-\phi_S)\sin i\phi}$	$i=1,2$		
7		2012	[46]	$A_{LU,\mathrm{I}}^{\sin i\phi}$	$i=1,2$	x_{Bj}	35 / 42
				$A_{LU,\mathrm{DVCS}}^{\sin i\phi}$	$i=1$		
				$A_{C}^{\cos i\phi}$	$i=0,1,2,3$		
8	CLAS	2001	[47]	$A_{LU}^{-,\sin i\phi}$	$i=1,2$	—	0 / 2
9		2006	[48]	$A_{UL}^{-,\sin i\phi}$	$i=1,2$	—	2 / 2
10		2008	[49]	A^{-}_{LU}		ϕ	283 / 737
11		2009	[50]	A^{-}_{LU}		ϕ	22 / 33
12		2015	[51]	A^{-}_{LU}, A^{-}_{UL}, A^{-}_{LL}		ϕ	311 / 497
13		2015	[52]	$d^4\sigma^{-}_{UU}$		ϕ	1333 / 1933
14	Hall A	2015	[34]	$\Delta d^4\sigma^{-}_{LU}$		ϕ	228 / 228
15		2017	[35]	$\Delta d^4\sigma^{-}_{LU}$		ϕ	276 / 358
16	COMPASS	2018	[36]	$d^3\sigma^{\pm}_{UU}$		t	2 / 4
17	ZEUS	2009	[37]	$d^3\sigma^{+}_{UU}$		t	4 / 4
18	H1	2005	[38]	$d^3\sigma^{+}_{UU}$		t	7 / 8
19		2009	[39]	$d^3\sigma^{\pm}_{UU}$		t	12 / 12
						SUM:	2624 / 3996

✍ Moutarde *et al.* (2019)

Almost all existing DVCS data sets.

2600+ measurements of 30 observables published during 2001-17.



▼ Hall A ● HERMES ■ COMPASS
▲ CLAS ◆ H1 and ZEUS

✍ Moutarde *et al.* (2019)

Compton Form Factors.

DVCS amplitude in the Bjorken regime.

Bjorken regime : large Q^2 and fixed $xB \simeq 2\xi/(1+\xi)$

- Partonic interpretation relies on **factorization theorems**.
- All-order proofs for DVCS.
- GPDs depend on a (arbitrary) factorization scale μ_F.
- **Consistency** requires the study of **different channels**.

- GPDs enter DVCS through **Compton Form Factors** :

$$\mathcal{F}(\xi, t, Q^2) = \int_{-1}^{1} \mathrm{d}x\, T\left(x, \xi, \alpha_S(\mu_F), \frac{Q}{\mu_F}\right) F(x, \xi, t, \mu_F)$$

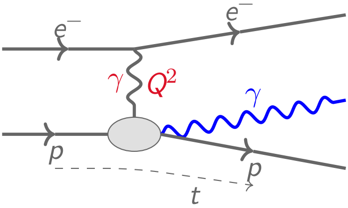
  for a given GPD F.
- Kernels T derived at NLO and (partially) NNLO.

✍ Belitsky and Müller (1998)

✍ Braun *et al.* (2022)

- CFF $\mathcal{F}$ is a **complex function**.

DVCS

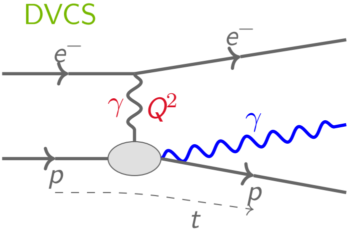


Compton Form Factors (CFF)

- Parametrize amplitudes.

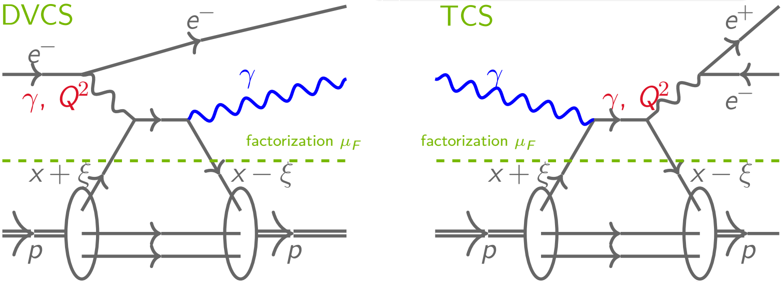
Compton scattering beyond leading order.

Scattering amplitudes and their partonic interpretation.



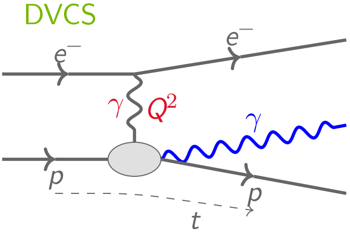
Compton Form Factors (CFF)

- Parametrize amplitudes.
- Evaluation at LO.



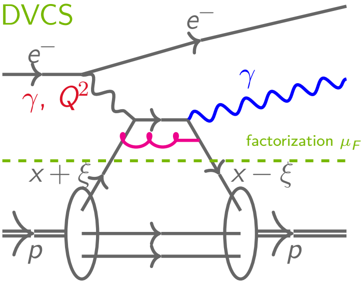
Compton scattering beyond leading order.

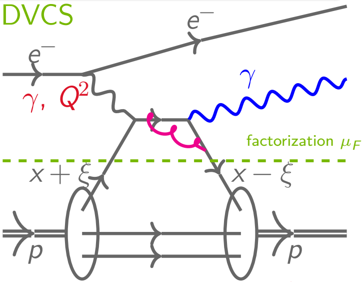
Scattering amplitudes and their partonic interpretation.

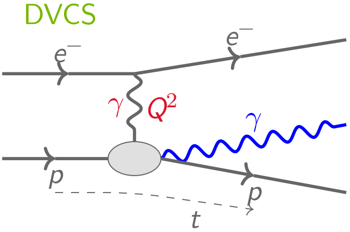


Compton Form Factors (CFF)

- Parametrize amplitudes.
- Evaluation at LO.
- Evaluation at NLO.

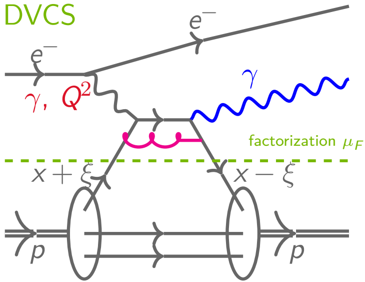


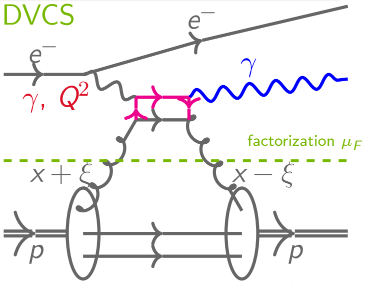




Compton Form Factors (CFF)

- Parametrize amplitudes.
- Evaluation at LO.
- Evaluation at NLO.
- Other diagrams at NLO, including gluon GPDs.





Explicit expressions of CFFs.

Quark and gluon contributions to $\mathcal{H}$ at LO and NLO (fixed t).

- Convolution of singlet GPD $H_q^+(x) \equiv H_q(x) - H_q(-x)$:

$$\mathcal{H}_q(\xi, Q^2) = \int_{-1}^{+1} dx\, H_q^+(x, \xi, \mu_F)\; T_q\left(x, \xi, \alpha_S(\mu_F), \frac{Q}{\mu_F}\right) + \int_{-1}^{+1} dx\, H_g(x, \xi, \mu_F)\; T_g\left(x, \xi, \alpha_S(\mu_F), \frac{Q}{\mu_F}\right)$$

✍ Belistky and Müller (1998)

✍ Pire *et al.* (2011)

- Convolution of singlet GPD $H_q^+(x) \equiv H_q(x) - H_q(-x)$:

$$\mathcal{H}_q(\xi, Q^2) \overset{\text{LO}}{=} \int_{-1}^{+1} dx\, H_q^+(x, \xi, \mu_F)\; C_0^q(x, \xi) + \int_{-1}^{+1} dx\, H_g(x, \xi, \mu_F)\; 0$$

✍ Belistky and Müller (1998)

✍ Pire *et al.* (2011)

- Integration yields **imaginary** parts to $\mathcal{H}$:

$$\mathrm{Im}\mathcal{H}_q(\xi, Q^2) \overset{\text{LO}}{=} \pi H_q^+(\xi, \xi, \mu_F)$$

- Convolution of singlet GPD $H_q^+(x) \equiv H_q(x) - H_q(-x)$:

$$\mathcal{H}_q(\xi, Q^2) \overset{\text{NLO}}{=} \int_{-1}^{+1} dx\, H_q^+(x, \xi, \mu_F) \left[C_0^q + C_1^q + \frac{1}{2} \ln \frac{|Q^2|}{\mu_F^2} C_{\text{Coll}}^q \right] + \int_{-1}^{+1} dx\, H_g(x, \xi, \mu_F) \left(0 + C_1^g + \frac{1}{2} \ln \frac{|Q^2|}{\mu_F^2} C_{\text{Coll}}^g \right)$$

✍ Belistky and Müller (1998)

✍ Pire *et al.* (2011)

- Integration yields **imaginary** parts to $\mathcal{H}$:

$$\text{Im}\mathcal{H}_q(\xi, Q^2) \overset{\text{NLO}}{=} \mathcal{I}(\xi) H_q^+(\xi, \xi, \mu_F) + \int_{-1}^{+1} dx\, \mathcal{T}^q(x) \Big(H_q^+(x, \xi, \mu_F) - H_q^+(\xi, \xi, \mu_F) \Big) + \text{gluon contributions.}$$

Imaginary part of Compton Form Factor $\mathcal{H}_q$ at NLO:

$$\mathrm{Im}\mathcal{H}_q(\xi, Q^2) \overset{\mathrm{NLO}}{=} \mathcal{I}(\xi) H_q^+(\xi, \xi, \mu_F) + \int_{-1}^{+1} dx \mathcal{T}^q(x) \Big(H_q^+(x, \xi, \mu_F) - H_q^+(\xi, \xi, \mu_F) \Big) + \text{gluon contributions.}$$

Due to $\mathcal{O}(\alpha_S(\mu_F))$ corrections:

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Due to $\mathcal{O}(\alpha_S(\mu_F))$ corrections:

- $\mathrm{Im}\mathcal{H}_q$ is **no more equal** to $\pi H_q^+(x = \xi, \xi)$ (LO):

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$$\mathrm{Im}\mathcal{H}_q(\xi, Q^2) \overset{\mathrm{NLO}}{=} \mathcal{I}(\xi) H_q^+(\xi, \xi, \mu_F) + \int_{-1}^{+1} dx \mathcal{T}^q(x) \Big(H_q^+(x, \xi, \mu_F) - H_q^+(\xi, \xi, \mu_F) \Big) + \text{gluon contributions.}$$

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  - Multiplicative factor $\mathcal{I}$ depends on ξ.

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  - $\mathrm{Im}\mathcal{H}_q$ **contains** gluon contributions.

Explicit expressions of CFFs.

Compton Scattering: LO vs NLO.

Imaginary part of Compton Form Factor $\mathcal{H}_q$ at NLO:

$$\mathrm{Im}\mathcal{H}_q(\xi, Q^2) \overset{\mathrm{NLO}}{=} \mathcal{I}(\xi) H_q^+(\xi, \xi, \mu_F) + \int_{-1}^{+1} dx\, \mathcal{T}^q(x) \Big(H_q^+(x, \xi, \mu_F) - H_q^+(\xi, \xi, \mu_F) \Big) + \text{gluon contributions.}$$

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  - $\mathrm{Im}\mathcal{H}_q$ **contains** gluon contributions.
- **No more direct link** to H_q even in valence region where $H_q(-\xi, \xi)$ is expected to be small.

Explicit expressions of CFFs.

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Due to $\mathcal{O}(\alpha_S(\mu_F))$ corrections:

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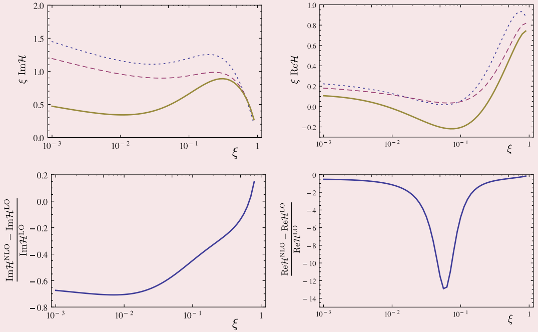
  - $\mathrm{Im}\mathcal{H}_q$ **contains** gluon contributions.
- **No more direct link** to H_q even in valence region where $H_q(-\xi, \xi)$ is expected to be small.

Question: What is the size of these $\mathcal{O}(\alpha_S(\mu_F))$ corrections?

NLO computations.

Large gluon contributions in some kinematic region.

$\mathcal{H}$ at LO and NLO ($t = -0.1$ GeV^2, $Q^2 = \mu_F^2 = 4.$ GeV^2)



✍ Moutarde *et al.* (2013)

dotted: LO   dashed: NLO quark corrections   solid: full NLO

- Using DDs, separate the D-term from the rest of the GPD

$$H(x,\xi,t,\mu_F) = \mathrm{sgn}(\xi) D\left(\frac{x}{\xi}, t, \mu_F\right) + \int_\Omega d\alpha d\beta\, \delta(x-\beta-\alpha\xi) f(\beta,\alpha,t,\mu_F)$$

- Read **analytic properties** of CFF $\mathcal{H}$ as a **function of** ξ

$$\mathcal{H}(\xi,t,Q^2) = \int_\Omega d\alpha d\beta\, T\left(\beta+\alpha\xi, \xi, \alpha_S(\mu_F), \frac{Q}{\mu_F}\right) f(\beta,\alpha,t,\mu_F)$$
$$+\text{similar D-term contribution}$$

from those of the DVCS coefficient function T.

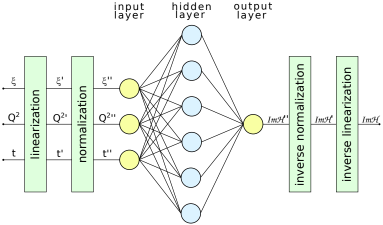
- Dispersion relations can be independently studied
  - at the level of the amplitude (CFF),
  - at the level of the coefficient function (factorization).
- The **subtraction constant** keeps track of the dominant singularity to apply Cauchy's theorem.
- **Factorization** relates this subtraction constant to the **D-term** through the DVCS coefficient function.

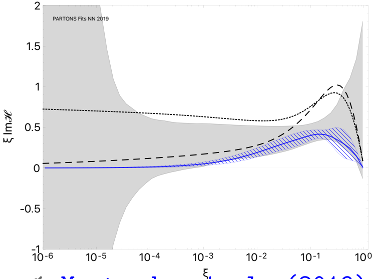
✍ Diehl & Ivanov (2007)

Once-subtracted dispersion relation for the CFF $\mathcal{H}$

$$\mathcal{C}_H(t, Q^2) = \mathrm{Re}\mathcal{H}(\xi, t, Q^2) + -\frac{1}{\pi}\int_0^1 \mathrm{d}\xi' \, \mathrm{Im}\mathcal{H}(\xi', t, Q^2)\left(\frac{1}{\xi - \xi'} - \frac{1}{\xi + \xi'}\right)$$

- Real and imaginary parts of CFFs parameterized by **neural networks**.
- Propagation of uncertainties through **replica method** and evaluation of 68 % **confidence levels**.





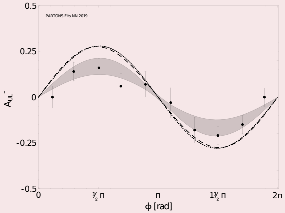
Moutarde *et al.* (2019)

A selection of results.

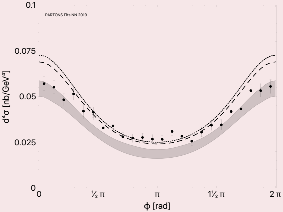
2600+ measurements of 30 observables published during 2001-17.

CEA - Saclay

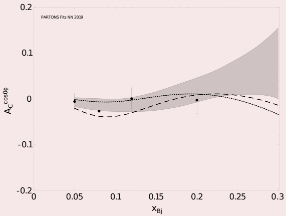
CLAS



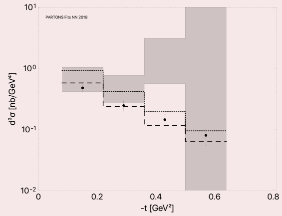
Hall A



HERMES

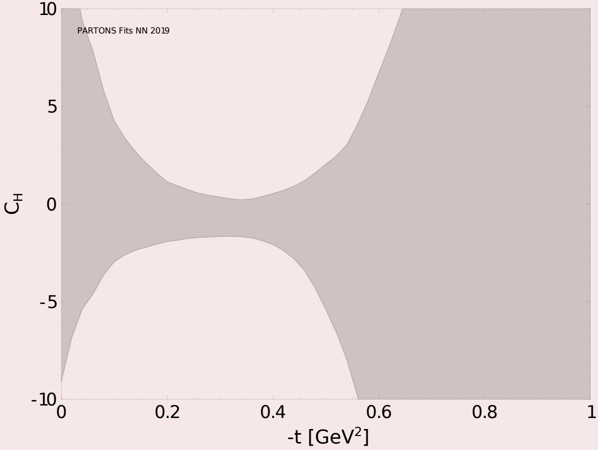


COMPASS



Moutarde *et al.* (2019)

Subtraction constant (related to pressure distribution)



Moutarde *et al.* (2019)

1 Expand D-term on Gegenbauer polynomials

$$D^q_{\mathrm{term}}(z, t, \mu_F^2) = (1 - z^2) \sum_{\text{odd } n} d^q_n(t, \mu_F^2)\, C_n^{3/2}(z)$$

2 Write dispersion relation for CFF (true at all pQCD orders)

$$\mathcal{C}_H(t, Q^2) = \mathrm{Re}\mathcal{H}(\xi) - \frac{1}{\pi}\int_0^1 \mathrm{d}\xi'\, \mathrm{Im}\mathcal{H}(\xi')\left(\frac{1}{\xi - \xi'} - \frac{1}{\xi + \xi'}\right)$$

3 Compute subtraction constant

$$\mathcal{C}_H^{q,g}(t, Q^2) = \frac{2}{\pi}\int_1^{+\infty} \mathrm{d}\omega\, \mathrm{Im}\, T^{q,g}(\omega) \int_{-1}^1 \mathrm{d}z\, \frac{D^{q,g}(z)}{\omega - z}$$

✍ Diehl & Ivanov (2007)

4 Retrieve GFF

$$d^q_1(t, \mu_F^2) = 5\, C_q(t, \mu_F^2)$$

Pressure forces from DVCS measurements.

A first-principle connection.

1. Expand D-term on Gegenbauer polynomials

$$D^q_{\mathrm{term}}(z, t, \mu_F^2) = (1 - z^2) \sum_{\text{odd } n} d_n^q(t, \mu_F^2)\, C_n^{3/2}(z)$$

2. Write dispersion relation for CFF (true at all pQCD orders)

$$\mathcal{C}_H(t, Q^2) = \mathrm{Re}\mathcal{H}(\xi) - \frac{1}{\pi}\int_0^1 \mathrm{d}\xi'\, \mathrm{Im}\mathcal{H}(\xi')\left(\frac{1}{\xi - \xi'} - \frac{1}{\xi + \xi'}\right)$$

3. Compute subtraction constant at LO

$$\mathcal{C}_H(t, Q^2) = 4 \sum_q e_q^2 \sum_{\text{odd } n} d_n^q(t, \mu_F^2 \equiv Q^2)$$

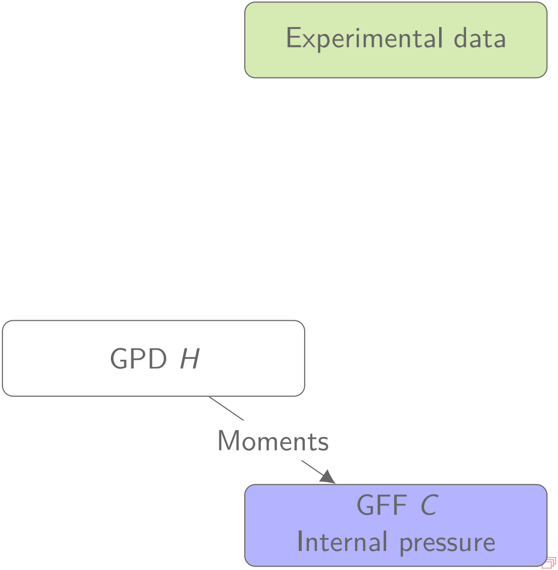
✍ Diehl & Ivanov (2007)

4. Retrieve GFF

$$d_1^q(t, \mu_F^2) = 5\, C_q(t, \mu_F^2)$$

Experimental data

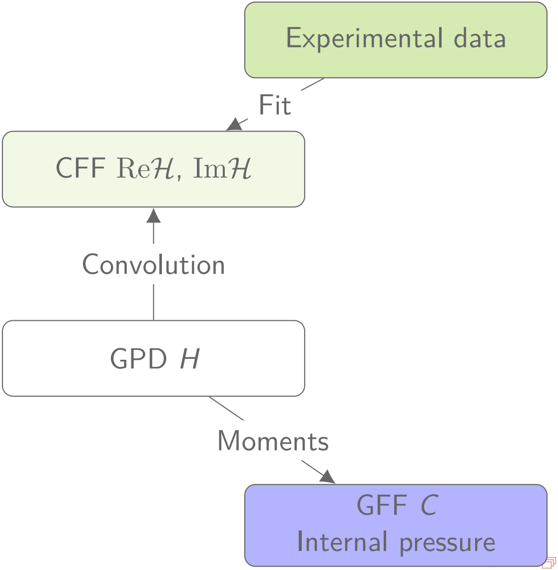
GFF C
Internal pressure



Experimental data
GPD H
Moments
GFF C
Internal pressure

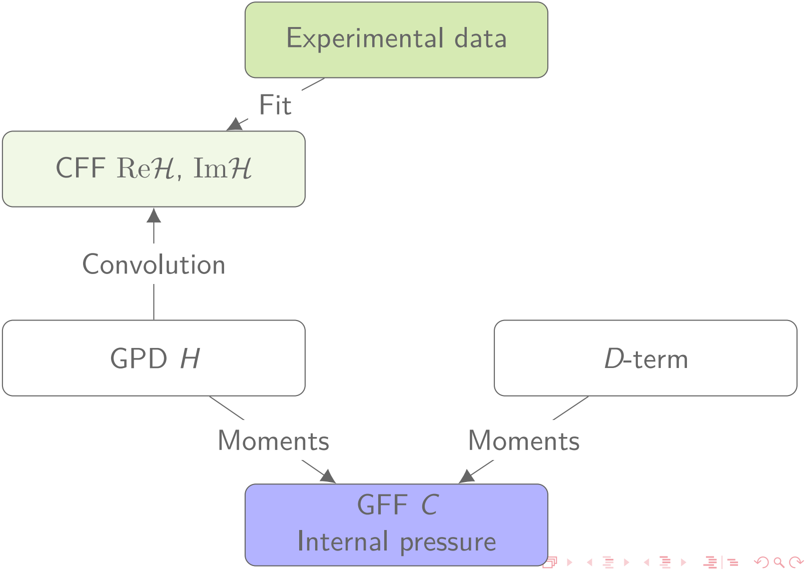
GFF extracting chain.

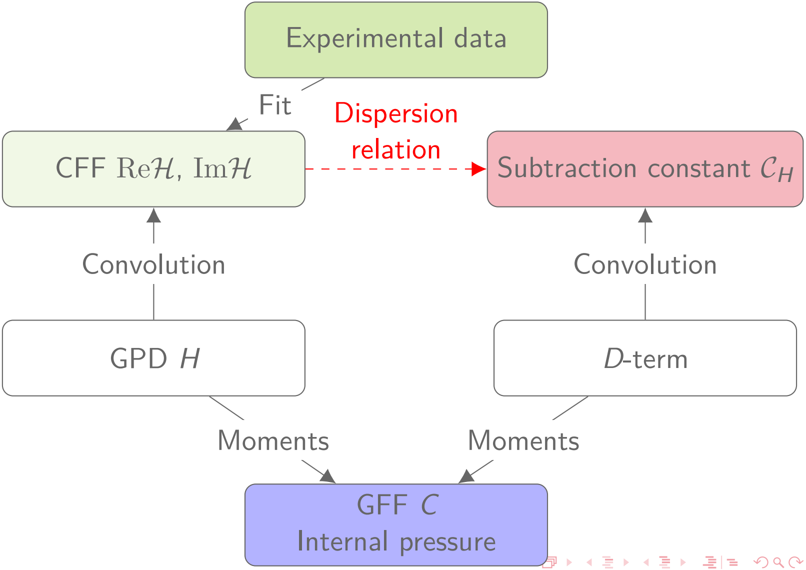
Implementing this first-principle connection.



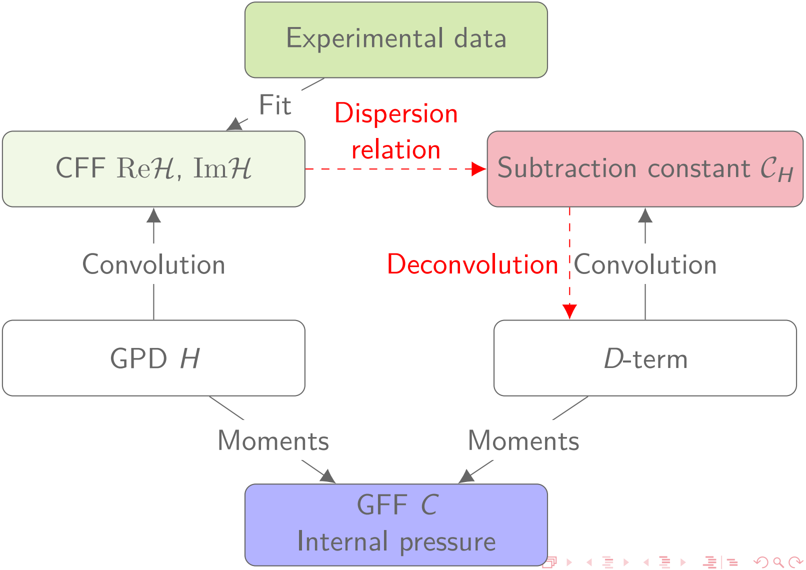
GFF extracting chain.

Implementing this first-principle connection.





Experimental data
Fit
Dispersion relation
CFF Re$\mathcal{H}$, Im$\mathcal{H}$
Subtraction constant $\mathcal{C}_H$
Convolution
Convolution
GPD H
D-term
Moments
Moments
GFF C
Internal pressure



Experimental data
Fit
Dispersion relation
CFF $\mathrm{Re}\mathcal{H}$, $\mathrm{Im}\mathcal{H}$
Subtraction constant $\mathcal{C}_H$
Convolution
Deconvolution
Convolution
GPD H
D-term
Moments
Moments
GFF C
Internal pressure

Subtraction constant from measurements.

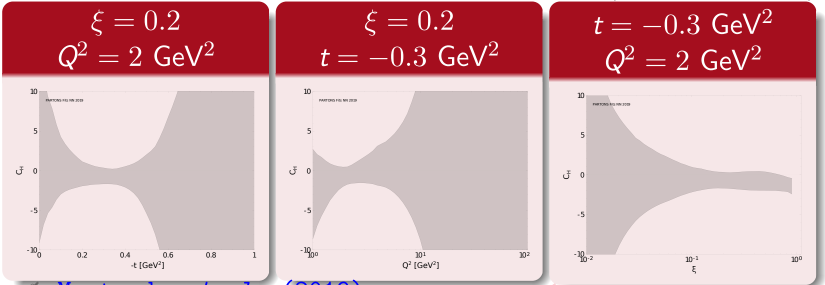
EIC prospect: determination over a wide kinematic domain.

- Range of kinematic variables in neural networks

$$\begin{array}{ccccc} 10^{-6} & < & \xi & < & 1 \\ 0 & < & -t & < & 1 \text{ GeV}^2 \\ 1 & < & Q^2 & < & 100 \text{ GeV}^2 \end{array}$$

- Implement DVCS dispersion relation

$$\mathcal{C}_H(t, Q^2) = \mathrm{Re}\mathcal{H}(\xi) - \frac{1}{\pi}\int_{10^{-6}}^{1} \mathrm{d}\xi' \, \mathrm{Im}\mathcal{H}(\xi) \left(\frac{1}{\xi - \xi'} - \frac{1}{\xi + \xi'} \right)$$



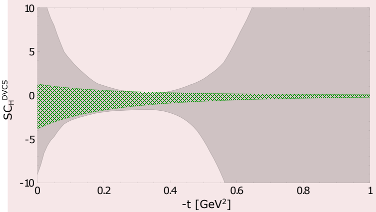
Moutarde *et al.* (2019)

Pressure forces from DVCS measurements.

Working assumptions.

1. Subtraction constant assumed equal to d_1.
2. Equal values for light quark contributions d_1^{uds}.
3. Radiative generation of gluon d_1^g and charm d_1^c contributions.
4. Tripole Ansatz $d_1(t, \mu_F) = d_1(\mu_F)(1 - t/\Lambda^2)^{-3}$.

Tripole Ansatz



Parameter	Value
$d_1^{uds}(\mu_F^2)$	-0.45 ± 0.92
$d_1^c(\mu_F^2)$	-0.0020 ± 0.0041
$d_1^g(\mu_F^2)$	-0.6 ± 1.3

✍ Dutrieux *et al.* (2021)

1. Subtraction constant assumed equal to d_1.
2. Equal values for light quark contributions d_1^{uds}.
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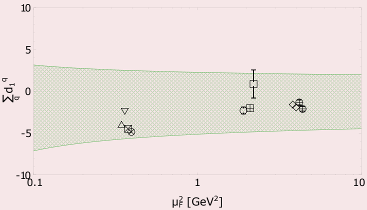
d_1 from DVCS data



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Dutrieux *et al.* (2021)

Pressure forces from DVCS measurements.

Working assumptions.

1. Subtraction constant assumed equal to d_1.
2. Equal values for light quark contributions d_1^{uds}.
3. Radiative generation of gluon d_1^g and charm d_1^c contributions.
4. Tripole Ansatz $d_1(t, \mu_F) = d_1(\mu_F)(1 - t/\Lambda^2)^{-3}$.

Summary of existing determinations

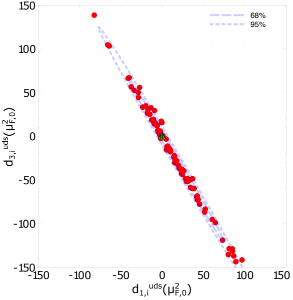
No.	Marker in Fig. 3	$\sum_q d_1^q(\mu_F^2)$	μ_F^2 in GeV²	# of flavours	Type	Ref.
1	○	$-2.30 \pm 0.16 \pm 0.37$	2.0	3	from experimental data	[13]
2	□	0.88 ± 1.69	2.2	2	from experimental data	[14]
3	◇	-1.59	4	2	t-channel saturated model	[55]
		-1.92	4	2	t-channel saturated model	[55]
4	△	-4	0.36	3	χQSM	[30]
5	▽	-2.35	0.36	2	χQSM	[10]
6	⊠	-4.48	0.36	2	Skyrme model	[56]
7	⊞	-2.02	2	3	LFWF model	[57]
8	⊗	-4.85	0.36	2	χQSM	[58]
9	⊕	-1.34 ± 0.31	4	2	lattice QCD ($\overline{\text{MS}}$)	[59]
		-2.11 ± 0.27	4	2	lattice QCD ($\overline{\text{MS}}$)	[59]

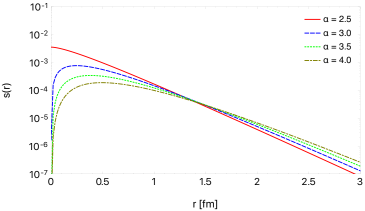
✍ Dutrieux *et al.* (2021)

From CFFs to nucleon mechanical structure.

A lot of model-dependence in current extractions.

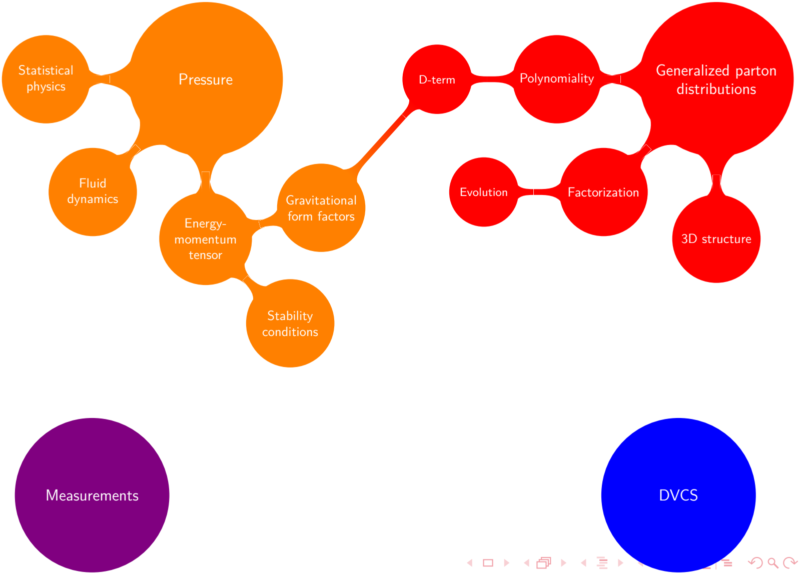
- **No justification to truncate the subtraction constant expansion to its first term** and assume that it is the d_1 coefficient related to the energy-momentum tensor.
- Shape of pressure profile is **fixed** by multipole Ansatz. Actual value is **extremely sensitive** to its parameters.





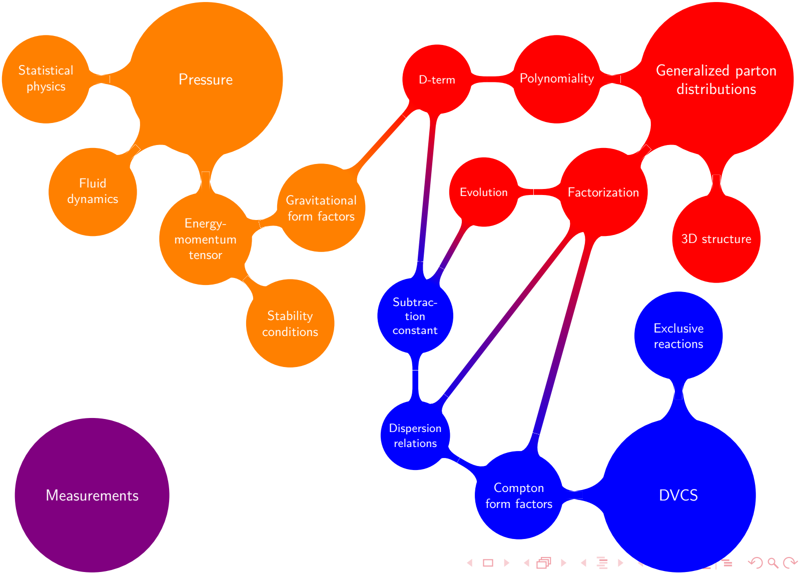
✍ Dutrieux *et al.* (2021)

Summary


Statistical physics
Pressure
Fluid dynamics
Energy-momentum tensor
Gravitational form factors
Stability conditions
D-term
Polynomiality
Generalized parton distributions
Evolution
Factorization
3D structure
Measurements
DVCS

What is the proton internal pressure?

Refining the concepts.



Abbreviations

Abbreviations used in this part.

ANN	artificial neural network
CFF	Compton form factor
DD	double distribution
DVCS	deeply virtual Compton scattering
DVMP	deeply virtual meson production
DR	dispersion relation
EIC	electron-ion collider
EFF	elastic form factor
GFF	gravitational form factor
GPD	generalized parton distribution
LO	leading order
NLO	next-to-leading order
TCS	timelike Compton scattering

Wednesday 7 Sep. 2022
10:00 - 11:00

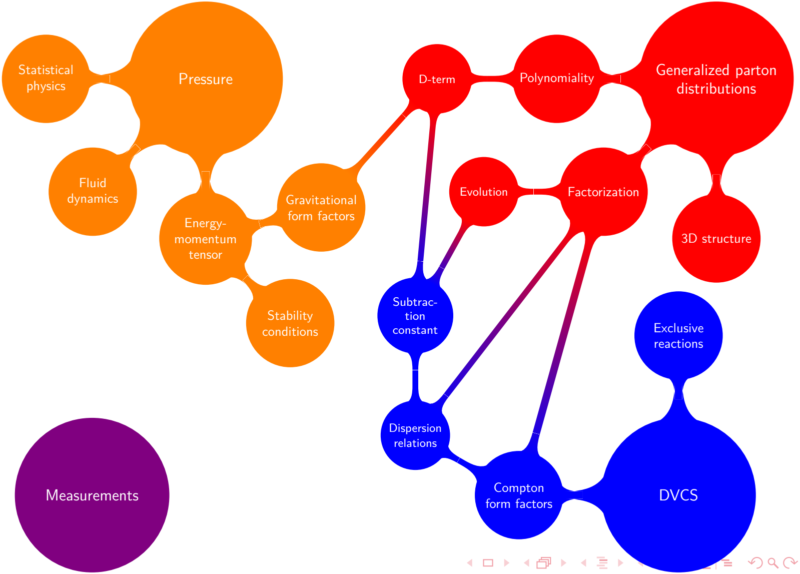
Part IV
Extraction of pressure distributions

From theory to numbers.

Go to outline.

What is the proton internal pressure?

Refining the concepts.



Increase the physics input in the global fit.

An example of the bias-variance trade-off.

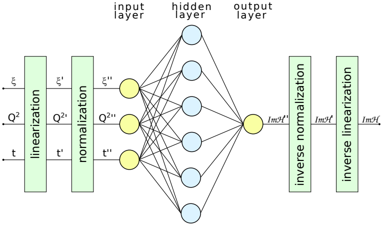
- So far the CFF fit gathering most of the world DVCS measurements relies on an **independent modeling** of the CFF real and imaginary parts by **neural networks**.
- Convenient because of the **dimensionality** of the problem but yields **large statistical uncertainties**.

  ✍ Moutarde *et al.* (2019)

- Conversaly a fit to the same data with a **physically motivated** parameterization still required ***ad hoc*** **assumptions**.

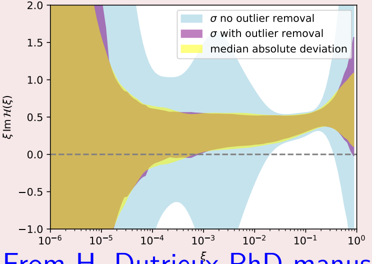
  ✍ Moutarde *et al.* (2018)

- Many **first-principle constraints** expressed at the GPD level are not implemented at the CFF level.

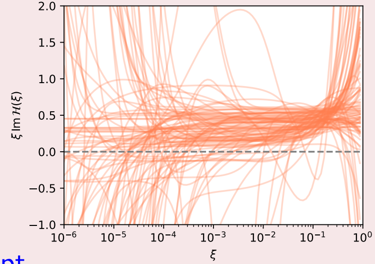


- Polynomiality?
- Positivity?
- Reduction to PDF or EFF?
- Evolution?

Outliers



Replicas

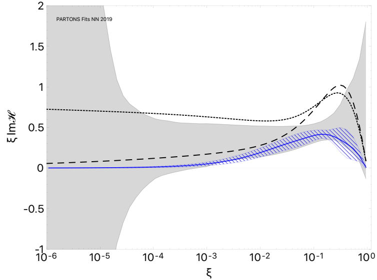


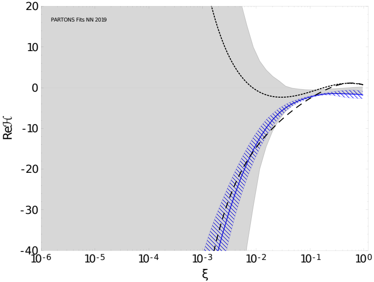
From H. Dutrieux PhD manuscript

Increase the physics input in the global fit.

An example of the bias-variance trade-off.

- Next step requires a (challenging) **GPD global fit** to world data.
- On the long run, need more experimental data to
  - Increase the Q^2-**lever arm**.
  - Provide a better handle on the real part of $\mathcal{H}$.
  - Improve the **accuracy** of existing measurements.
  - Probe the kinematic regions insufficiently constrained.





Moutarde *et al.* (2019)

Relax modeling assumptions on $d_1(t)$.

Shape of pressure distribution not set by the current fit...

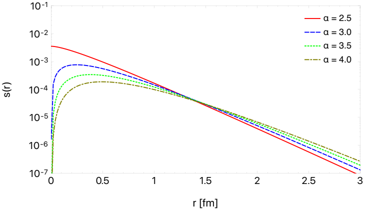
- Use **multipole Ansatz**

$$d_1(t, \mu_F) = \frac{d_1(\mu_F)}{\left(1 - \frac{t}{\Lambda^2}\right)^\alpha}$$

- Remind $d_1^q(t, \mu_F^2) = 5C_q(t, \mu_F^2)$.
- Plug in pressure anisotropy

$$\frac{s(r)}{M} \propto \int \frac{\mathrm{d}^3\vec{\Delta}}{(2\pi)^3} e^{-i\vec{\Delta}\cdot\vec{r}} \left\{ -\frac{4}{r^2}\frac{t^{-1/2}}{M^2}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(t^{5/2}\, d_1(t)\right) \right\}$$

- Normalization $d_1(\mu_F)$ set by fit.
- Position of node in r depends on Λ.



✍ Dutrieux *et al.* (2021)

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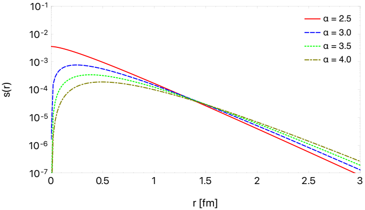


✍ Dutrieux *et al.* (2021)

- **Asymptotic** information on $|t|$-dependence from perturbative QCD. *But how large is "asymptotic"?*
- **Factorization** constraint: $Q^2 \gg |t|$. *Most of the experimental data used as fit input has low $|t|$.*
- Need for more experimental data points.

Increase Q^2-lever arm.

Evolution equations bring a slow $\log Q^2$ dependence.

- Remind computation of subtraction constant at LO

$$\mathcal{C}_H(t, Q^2) = 4 \sum_q e_q^2 \sum_{\text{odd } n} d_n^q(t, \mu_F^2 \equiv Q^2)$$

✍ Diehl & Ivanov (2007)

- Plug LO evolution of D-term to obtain the following **pattern**

$$\mathcal{C}_H(t, Q^2) \propto \sum_{\text{odd } n} d_n(t, \mu_F) \left(\frac{\alpha_s(Q^2)}{\alpha_s(\mu_F^2)} \right)^{\gamma_n}$$

with γ_n computed in perturbative QCD.

- Since $\alpha_s(Q^2) \propto 1/\log Q^2$, an exact knowledge of $\mathcal{C}_H(t, Q^2)$ on an Q^2-interval allows to exactly retrieve d_n.

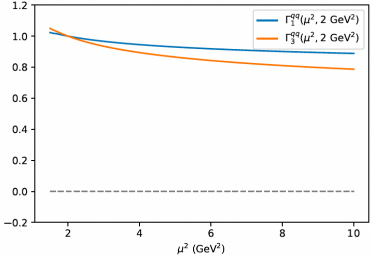
Increase Q^2-lever arm.

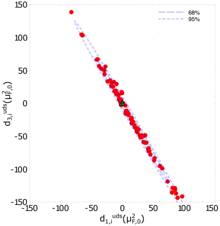
Anomalous dimensions γ_n are small and take comparable values.

- Introduce evolution operator Γ so that

$$d_n(\mu_1) = \Gamma_n(\mu_1, \mu_2) d_n(\mu_2)$$

- Probed Q^2-range in CFF fit: $[1.5, 4]$ GeV^2.
- Γ_1 and Γ_3 are **numerically very close**.





- Evolution **"too slow"** to separate d_1 and d_3 for $Q^2 \in [1.5, 4]$ GeV^2.
- Experimental data mostly constrain $d_1 + d_3 + \ldots$

✍ Dutrieux *et al.* (2021)

Anomalous dimensions γ_n and Q^2-lever arm.

Inverse problem and regularization.

- Remind **pattern** of the problem

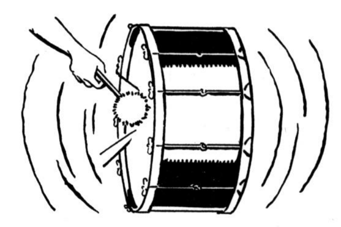
$$\mathcal{C}_H(t, Q^2) \propto \sum_{\text{odd } n} d_n(t, \mu_F) \left(\frac{\alpha_s(Q^2)}{\alpha_s(\mu_F^2)} \right)^{\gamma_n}$$

- If Q^2-range is too small, a solution with $d_1(t, \mu_F) + d_3(t, \mu_F) + d_5(t, \mu_F) + \ldots = 0$ can remain **hidden within experimental uncertainties** over the whole range $Q^2 \in [Q^2_{\text{min}}, Q^2_{\text{max}}]$.
- In practice: act as if the problem of retrieving d_1, $d_3, \ldots$ from measurements has infinitely many solutions.
- Add extra **regularization** to select one solution **robust with respect to statistical uncertainties**.
- Today **cannot reliably estimate** the uncertainty associated to the neglect of $d_3, \ldots$

"Can one hear the shape of a drum?"

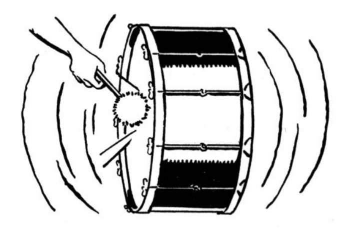
✍ Kac (1966)

"Can one hear the shape of a drum?"



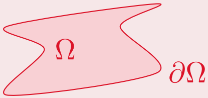
✍ Kac (1966)

"Can one hear the shape of a drum?"



✍ Kac (1966)

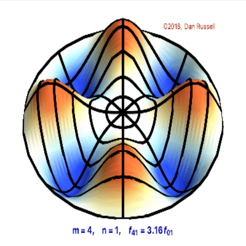
In quantitative terms



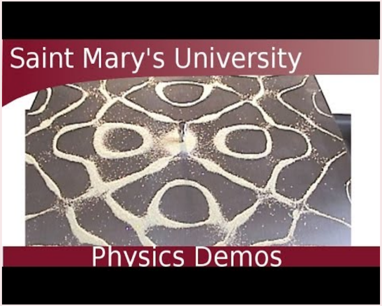
- Dirichlet problem for the Laplacian:

$$\Delta u + \lambda u = 0 \quad \text{and} \quad u_{|\partial\Omega} = 0$$

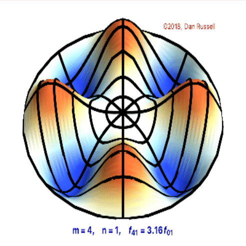
Vibration patterns



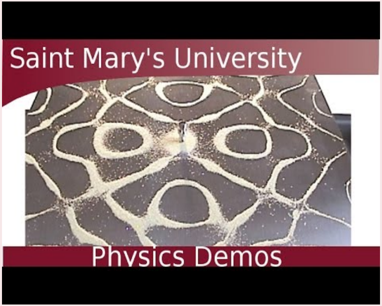
Vibration patterns



Vibration patterns



Vibration patterns



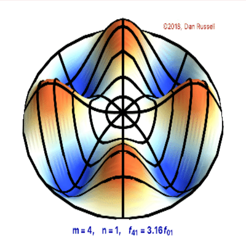
Analogy: what about the proton?

- "Hit" the proton, *e.g.* with a virtual photon:
- "Listen" to the distribution of produced particles:
- "Measure" harmonics:

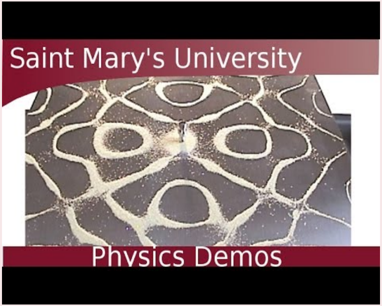
What is an inverse problem?

Harmonics and patterns.

Vibration patterns



Vibration patterns



Analogy: what about the proton?

- "Hit" the proton, *e.g.* with a virtual photon: hard
- "Listen" to the distribution of produced particles: exclusive
- "Measure" harmonics: GPDs or CFFs

- Assume CFF $\mathcal{H}$ is perfectly known. Solve inverse problem?

$$\mathcal{H}^q(\xi, Q^2) = \int_{-1}^{1} \frac{\mathrm{d}x}{\xi} T^q\left(\frac{x}{\xi}, \frac{Q^2}{\mu^2}, \alpha_s(\mu^2)\right) H^q(x, \xi, \mu^2)$$

- Question raised about 20 years ago and has remained essentially open. Evolution proposed as a crucial element.

✍ Freund (2000)

- There exist **non-zero** GPDs with **vanishing forward limit** and **vanishing CFF** up to order α_s^2.
- The DVCS deconvolution problem is **ill-posed**.

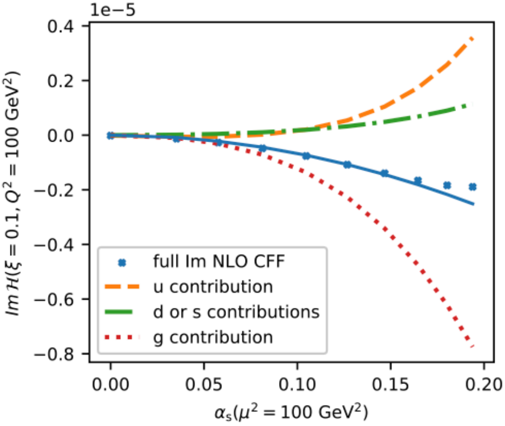
✍ Bertone *et al.* (2021)

- **Same conclusion holds** for several other hard exclusive processes.
- **Define** and **implement** further criterions in fitting strategies to select one solution among infinitely many.

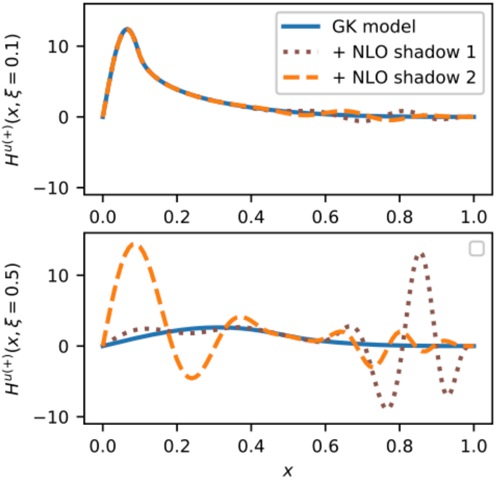
From CFFs to GPDs.

Shadow GPDs have null LO and NLO CFF.

- Start with shadow GPD for flavor u at 1 GeV^2.
- Generate d, s and g while evolving up to 100 GeV^2.
- Compute resulting CFF.



✍ Bertone *et al.* (2021)



Bertone *et al.* (2021)

Energy and mechanical radii.

New notions of "nucleon size" beyond electric radius.

- Reminder

$$\frac{\varepsilon_a(r)}{M} = \int \frac{\mathrm{d}^3\vec{\Delta}}{(2\pi)^3}\, e^{-i\vec{\Delta}\cdot\vec{r}} \left\{ A_a(t) + \bar{C}_a(t) + \frac{t}{4M^2}\left[B_a(t) - 4C_a(t)\right] \right\}$$

$$\frac{p_{r,a}(r)}{M} = \int \frac{\mathrm{d}^3\vec{\Delta}}{(2\pi)^3}\, e^{-i\vec{\Delta}\cdot\vec{r}} \left\{ -\bar{C}_a(t) - \frac{4}{r^2}\frac{t^{-1/2}}{M^2}\frac{\mathrm{d}}{\mathrm{d}t}\left(t^{3/2}\,C_a(t)\right) \right\}$$

- Define **energy** and **mechanical** radii

$$\langle r^2\rangle_E = \frac{1}{M}\int \mathrm{d}^3\vec{r}\, r^2 \epsilon(r)$$

$$\langle r^2\rangle_{\mathrm{mech}} = \frac{1}{\mathcal{P}_r}\int \mathrm{d}^3\vec{r}\, p_r(r)$$

with $\mathcal{P}_r = \int \mathrm{d}^3\vec{r}\, r^2 p_r(r)$.

✍ Polyakov and Schweitzer (2018)

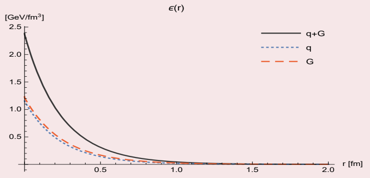
✍ Lorcé *et al.* (2019)

Equation of state.

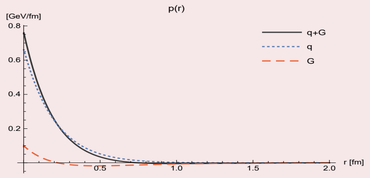
Elaborating on the relation between energy and pressure.

- Simple multiple models: dipole for GFFs A and $\bar{C}$, tripole for GFFs B and C.

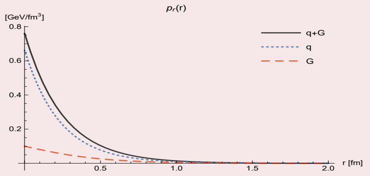
quark



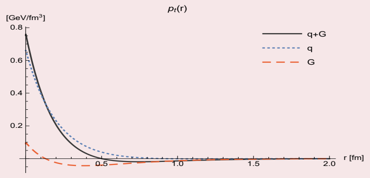
glue



quark + glue



Neutron stars

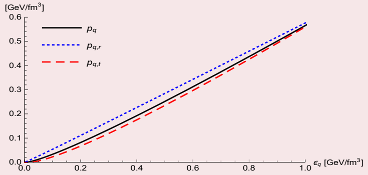


Lorcé *et al.* (2019)

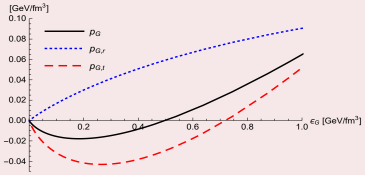
Equation of state.

Elaborating on the relation between energy and pressure.

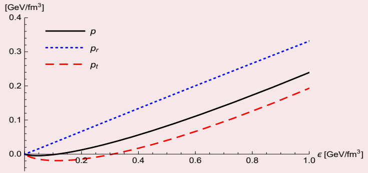
quark



glue



quark + glue

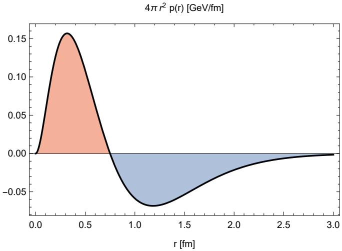


✍ Lorcé *et al.* (2019)

- Parametric plots of EOS
  - $(\epsilon(r), p_r(r))$
  - $(\epsilon(r), p_t(r))$
  - $(\epsilon(r), p(r))$
- Quark and gluon contributions

- Conservation of total EMT $\partial_\mu T^{\mu\nu} = 0$ implies in the Breit frame

$$\frac{\mathrm{d}p_r(r)}{\mathrm{d}r} = -\frac{2s(r)}{r}$$



✍ Lorcé *et al.* (2019)

- Consequence: von Laue condition

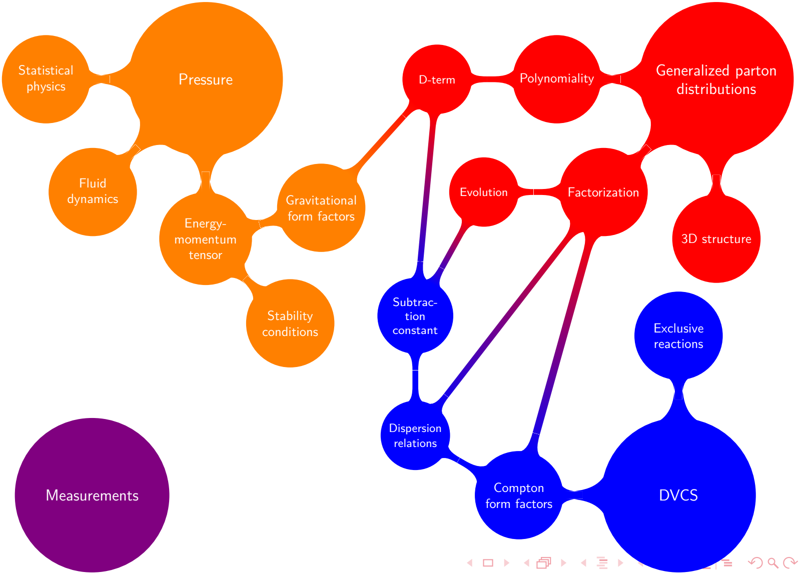
$$\int_0^\infty \mathrm{d}r\, r^2 p(r) = 0$$

Stability in hydrodynamics.

Consequences on GFFs?

- Expectations for stable systems
  - $\epsilon(0) < \infty$, $p(0) < \infty$ and $s(0) = 0$.
  - $\epsilon(r) > 0$ and $p_r(r) > 0$
  - $\mathrm{d}\epsilon/\mathrm{d}r < 0$ and $\mathrm{d}p_r/\mathrm{d}r < 0$
- Conjecture: $C(0) < 0$ (and so is d_1).
- Phenomenological or theoretical checks: other theories or targets (not just hadrons).
- Further characterization of the underlying dynamics?

Summary



Statistical physics
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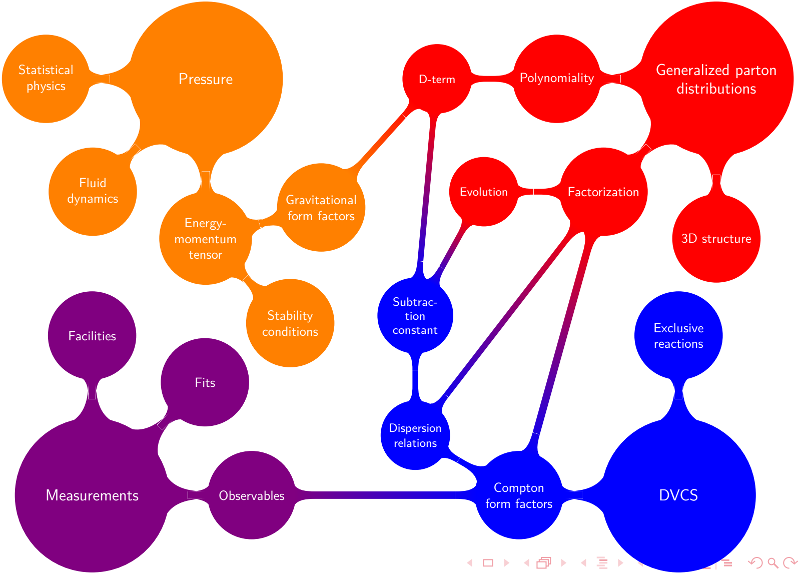
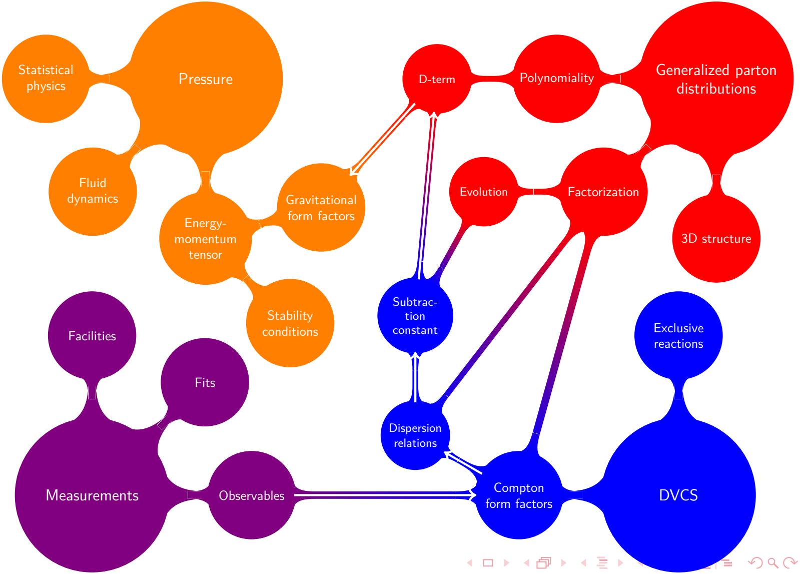
3D structure
Subtrac-tion constant
Exclusive reactions
Dispersion relations
Compton form factors
DVCS
Measurements

What is the proton internal pressure?
Refining the concepts.



What is the proton internal pressure?

Refining the concepts.



Abbreviations

Abbreviations used in this part.

ANN	artificial neural network
CFF	Compton form factor
DDVCS	double deeply virtual Compton scattering
DVCS	deeply virtual Compton scattering
DVMP	deeply virtual meson production
DR	dispersion relation
EIC	electron-ion collider
EFF	elastic form factor
GFF	gravitational form factor
GPD	generalized parton distribution
LO	leading order
NLO	next-to-leading order
PDF	parton distribution function
TCS	timelike Compton scattering

Conclusion and prospects

- Concept **well-defined** and suitable for phenomenology.
- Strong **first-principle connection** between concept and experimental data.
- Need for **multi-channel** analysis **beyond LO** on a **wide kinematic coverage**. EIC much needed!
- The GPD deconvolution problem is **ill-posed**. **Huge sensitivity** to numerical noise or experimental uncertainties.
- Development of a **software ecosystem** for 3D hadron structure studies.
- Need for **coordinated effort** involving fits, computing chains *e.g.* `PARTONS` and lattice QCD to make the best from experiments.

gg75478317 GoGraph.com

Thank you for your attention!

Contact herve.moutarde@cea.fr for post-EJC 2022 questions.

Appendix

- From **uncertainty principle**: minimal spread in momentum / energy in a **confined** system.

Charge radius: fully relativistic treatment.

Localization of a quantum relativistic system.

- From **uncertainty principle**: minimal spread in momentum / energy in a **confined** system.
- If the energy levels of the confined system are high enough, **pair creation** is possible.

Charge radius: fully relativistic treatment.

Localization of a quantum relativistic system.

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- Pair creation may **prevent the localization** of a particle with a high resolution.

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Localization of a quantum relativistic system.

- From **uncertainty principle**: minimal spread in momentum / energy in a **confined** system.
- If the energy levels of the confined system are high enough, **pair creation** is possible.
- Pair creation may **prevent the localization** of a particle with a high resolution.
- Discussions about nucleon radius refers to a **specific prescription**.

- **Wave packet** for spinless mass m particle localized at $\vec{R}$:

$$\left|\vec{R}\right\rangle = \int \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} e^{i\vec{p}\cdot\vec{R}} \psi(\vec{p}) \left|\vec{p}\right\rangle \text{ with } E_p = \sqrt{\vec{p}^2 + m^2}$$

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- Reminder: Definition of form factor

$$\left\langle p' \left| J^{\mathrm{e.m.}}_\mu(0) \right| p \right\rangle = (p_\mu + p'_\mu) F(q^2)$$

- Fourier transform of **charge distribution**:

$$\int \mathrm{d}^3\vec{r} e^{i\vec{q}\cdot\vec{r}} \left\langle \vec{R} \middle| \rho(\vec{r}) \middle| \vec{R} \right\rangle = \int \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3} \frac{\psi^*(\vec{p}+\vec{q})\psi(\vec{p})}{\sqrt{E_p E_{p+q}}} \left\langle \vec{p}\,' \middle| \rho(\vec{0}) \middle| \vec{p} \right\rangle$$

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- Three types of contributions
  Form factor **sensitivity** form factor's shape: cannot take F out of the integral.
  $$q^0 = \sqrt{(\vec{p}+\vec{q})^2 + M^2} - \sqrt{\vec{p}^2 + M^2}$$

Burkardt (2000)

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Wave packet Sensitivity to **spatial distribution** of the wave packet.

Burkardt (2000)

- 3D Fourier transform of charge distribution:

$$\int \mathrm{d}^3\vec{r}\, e^{i\vec{q}\cdot\vec{r}} \left\langle \vec{R} | \rho(\vec{r}) | \vec{R} \right\rangle = \int \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3} \frac{E_p + E_{p+q}}{2\sqrt{E_p E_{p+q}}} \psi^*(\vec{p}+\vec{q})\psi(\vec{p}) F(q^2)$$

- Three types of contributions
  - Form factor: **sensitivity** form factor's shape: cannot take F out of the integral.
  - Wave packet: Sensitivity to **spatial distribution** of the wave packet.
  - Relativistic effects: Nonrelativistic limit $\vec{p}^2 \ll m^2$:

$$E_p \simeq m + \frac{\vec{p}^2}{2m} \text{ and } \frac{E_p + E_{p+q}}{2\sqrt{E_p E_{p+q}}} \simeq 1$$

Burkardt (2000)

- 3D Fourier transform of charge distribution:

$$\int d^3\vec{r}\, e^{i\vec{q}\cdot\vec{r}} \left\langle \vec{R} | \rho(\vec{r}) | \vec{R} \right\rangle = \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{E_p + E_{p+q}}{2\sqrt{E_p E_{p+q}}} \psi^*(\vec{p}+\vec{q})\psi(\vec{p}) F(q^2)$$

- Three types of contributions
  - Form factor: **sensitivity** form factor's shape: cannot take F out of the integral.
  - Wave packet: Sensitivity to **spatial distribution** of the wave packet.
  - Relativistic effects: Nonrelativistic limit $\vec{p}^2 \ll m^2$:
- 3D Fourier transform of charge distribution is F when:
  - Wave packet is very broad in momentum space.
  - Nonrelativistic limit.

✍ Burkardt (2000)

Charge radius: fully relativistic treatment.

Quantum relativistic localization.

- Expand 3D Fourier transform of charge distribution:

$$\int \mathrm{d}^3\vec{r}\, e^{i\vec{q}\cdot\vec{r}} \left\langle \vec{R} | \rho(\vec{r}) | \vec{R} \right\rangle = \int \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3} \frac{E_p + E_{p+q}}{2\sqrt{E_p E_{p+q}}} \psi^*(\vec{p}+\vec{q})\psi(\vec{p}) F(q^2)$$

$$\simeq 1 + \frac{\langle r^2 \rangle}{6}\vec{q}^2 - \frac{\langle r^2 \rangle}{6} \int \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3} |\psi(\vec{p})|^2 \frac{(\vec{q}\cdot\vec{p})^2}{E_p^2}$$

$$+ \int \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3} |\vec{q}\cdot\nabla\psi(\vec{p})|^2 - \frac{1}{8} \int \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3} |\psi(\vec{p})|^2 \frac{(\vec{q}\cdot\vec{p})^2}{E_p^4}$$

- **Relativistic corrections** appear with terms $\propto (\vec{q}\cdot\vec{p})^2/E_p^2$ or $\vec{q}^2/E_p^2$.
- In a reference frame where E_p is large and $\vec{q}^2$ and $\vec{p}\cdot\vec{q}$ are finite, these **corrections remain small**.

✍ Burkardt (2000)

Charge radius: fully relativistic treatment.

Quantum relativistic localization in an infinite momentum frame.

- Reference frame with a **fast moving** particle along z axis:

$$p_\mu \simeq \left(P + \frac{m^2}{2P}, 0_\perp, P \right) \text{ for large } P$$

- In the **Bjorken frame** the 4-momentum of the exchanged photon is:

$$q_\mu = \left(\frac{Q^2}{2x_B P}, q_\perp, 0 \right)$$

- With this choice are kept **finite** when $P \to \infty$:

$$p \cdot q = \frac{Q^2}{2x_B} + \frac{m^2 Q^2}{4x_B P^2} \text{ and } q^2 = \left(\frac{Q^2}{2x_B P} \right)^2 - q_\perp^2$$

- In that frame the wave packet in completely **delocalized** in z direction and **sharply peaked** in transverse directions.
- Consistent relativistic def.: form factor $\equiv$ **2D Fourier transform** of charge distribution in **transverse plane**.

- The Poincaré group is defined by:
  - 4 **translation** generators P^μ
  - 3 **spatial rotation** generators J^i
  - 3 **boost** generators K^i
- The 6 light-cone generators J^3, P^1, P^2, P^+, $(K^1 + J^2)/\sqrt{2}$, and $(K^2 - J^1)/\sqrt{2}$ leave **invariant** the surfaces of constant x^+.
- P^- generates translations in x^+ directions: **Hamiltonian**.
- The sub-algebra generated by these 7 generators is **isomorphic** to the algebra of **Galilean transformations** of 2D quantum mechanics:

$$\begin{array}{lcl} P^+ & \leftrightarrow & \text{Mass} \\ P^- & \leftrightarrow & \text{Hamiltonian} \\ J^3 & \leftrightarrow & \text{Rotations in transverse plane} \\ P^\perp & \leftrightarrow & \text{Translations in transverse plane} \end{array}$$

Wigner operator for quarks at fixed light-cone time $y^+ = 0$

$$\hat{\mathcal{W}}^q_\Gamma(\vec{b}_\perp, \vec{k}_\perp, x) =$$

$$\frac{1}{2}\int \frac{\mathrm{d}z^- \mathrm{d}^2 z_\perp}{(2\pi)^3} e^{i(xP^+ z^- - \vec{k}_\perp \cdot \vec{z}_\perp)} \bar{q}\left(y - \frac{z}{2}\right)\Gamma\mathcal{L}q\left(y + \frac{z}{2}\right)\Big|_{z^+=0}$$

where:

✍ Lorcé and Pasquini (2011)

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where:

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Quark Wigner distribution.
Full relativistic treatment.

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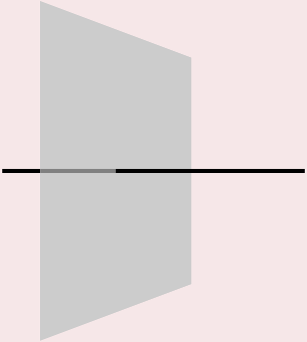

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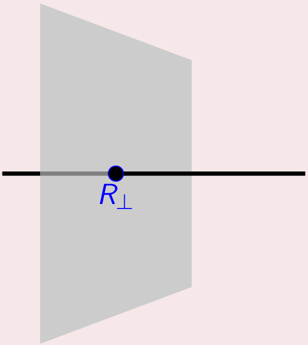

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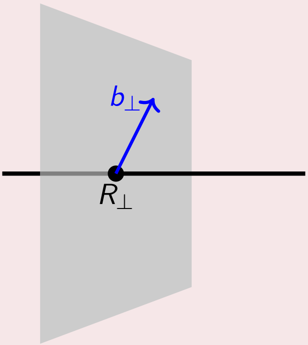

- Transverse center of momentum $R_\perp = \sum_i x_i r_{\perp i}$,
- Impact parameter $b_\perp$,

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$$\hat{\mathcal{W}}^q_\Gamma(\vec{b}_\perp, \vec{k}_\perp, x) =$$

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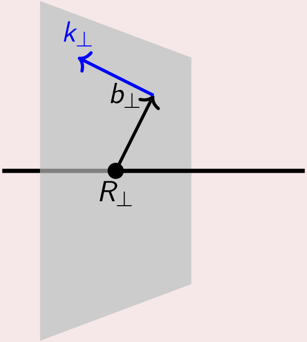

- Transverse center of momentum $R_\perp = \sum_i x_i r_{\perp i}$,
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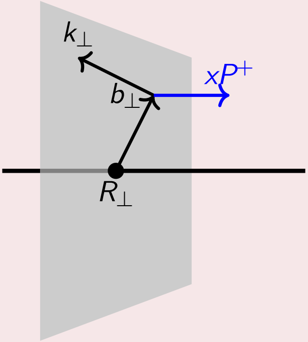

- Transverse center of momentum $R_\perp = \sum_i x_i r_{\perp i}$,
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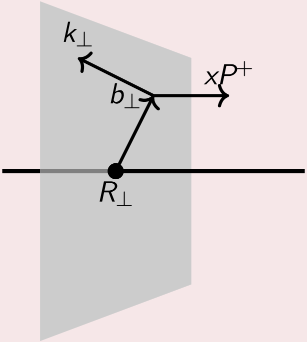


- Transverse center of momentum $R_\perp = \sum_i x_i r_{\perp i}$,
- Impact parameter $b_\perp$,
- Transverse momentum $k_\perp$,
- Longitudinal momentum xP^+.

Quark Wigner distribution.

Wigner distributions as matrix elements of localized nucleon states.

- Take a nucleon state $\left| p^+, \vec{p}_\perp, \vec{S} \right\rangle$ where $\vec{S}$ is the **polarization** of the nucleon.

Wigner distribution (quantum relativistic framework)

$$\mathcal{W}^q_\Gamma(\vec{b}_\perp, \vec{k}_\perp, x, \vec{S}) \equiv$$
$$\int \frac{\mathrm{d}^2\Delta_\perp}{(2\pi)^2} \left\langle p^+, \frac{\Delta_\perp}{2}, \vec{S} \middle| \hat{\mathcal{W}}^q_\Gamma(\vec{b}_\perp, \vec{k}_\perp, x) \middle| p^+, -\frac{\Delta_\perp}{2}, \vec{S} \right\rangle$$

- Wigner distributions are **2D Fourier transforms** of more general objects: GTMDs.
- Leading twist: 16 GTMDs (complex-valued functions).

✍ Meissner *et al.* (2009)

✍ Meissner *et al.* (2008)

- Thus there are 16 Wigner distributions which are real-valued functions (leading twist).

$x, \xi, \vec{k}_\perp, \vec{\Delta}_\perp$ **GTMD**

$$\xi = -\frac{\Delta^+}{2P^+}$$

$$\Delta^2 = -\frac{4\xi^2 M^2 + \vec{\Delta}_\perp^2}{1-\xi^2}$$

Back to Wigner distributions.

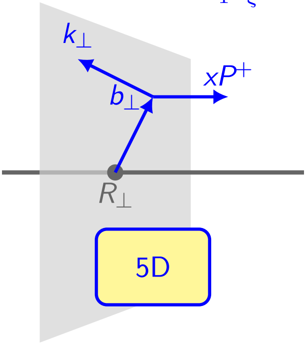
The family of 1-quark distributions.

GPDs and TMDs provide complementary 3D information.

$x, \xi, \vec{k}_\perp, \vec{\Delta}_\perp$ GTMD → Fourier $\Delta_\perp \leftrightarrow b_\perp$, $\xi = 0$ → Wigner $x, \vec{k}_\perp, \vec{b}_\perp$

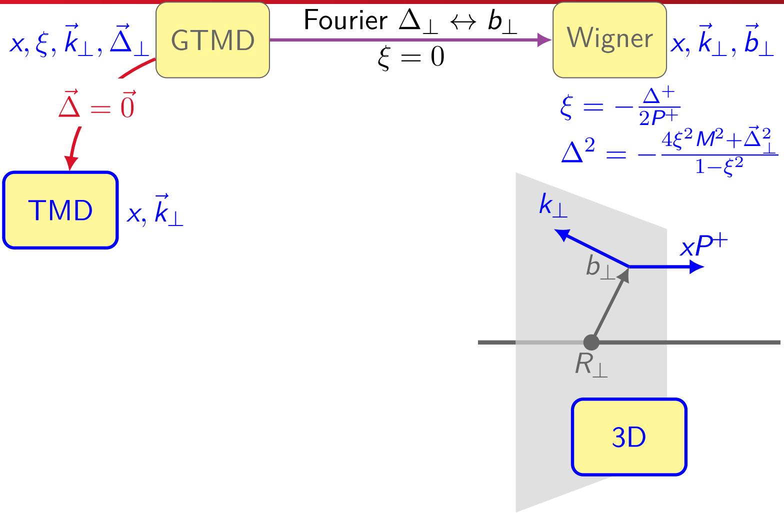
$$\xi = -\frac{\Delta^+}{2P^+}$$

$$\Delta^2 = -\frac{4\xi^2 M^2 + \vec{\Delta}_\perp^2}{1-\xi^2}$$



The family of 1-quark distributions.

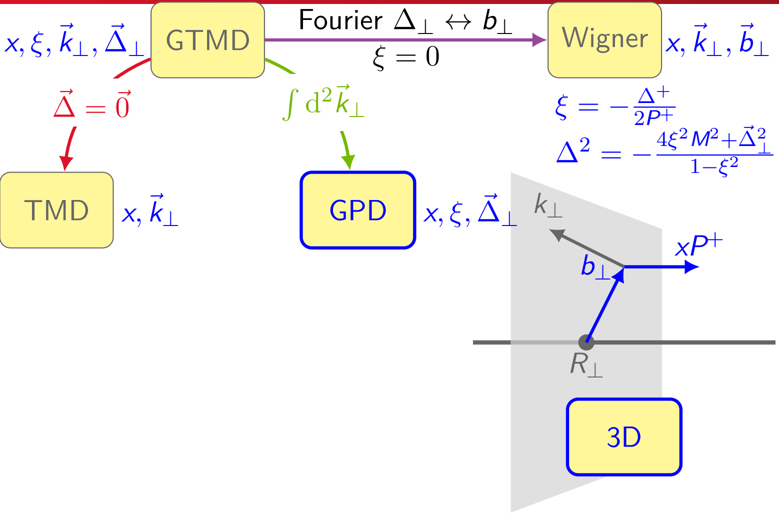
GPDs and TMDs provide complementary 3D information.



Back to Wigner distributions.

The family of 1-quark distributions.

GPDs and TMDs provide complementary 3D information.



Back to Wigner distributions.

The family of 1-quark distributions.

GPDs and TMDs provide complementary 3D information.

Exclusive reactions as a nuclear manometer

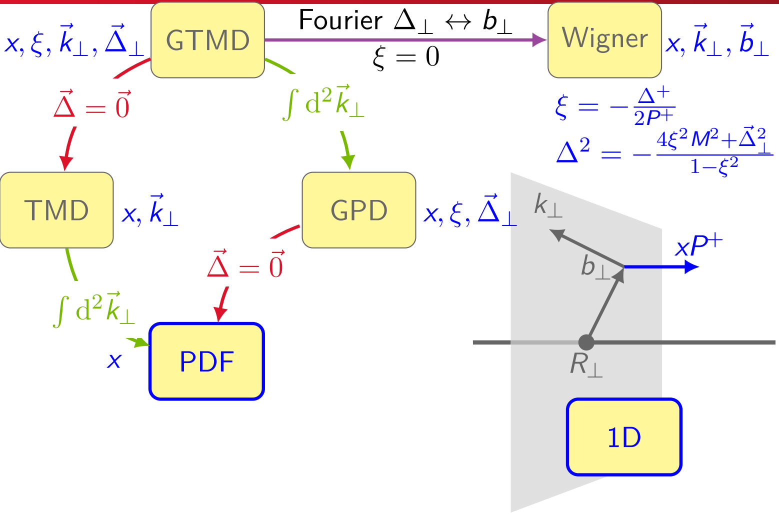
Quark Wigner distributions

Relativistic treatment

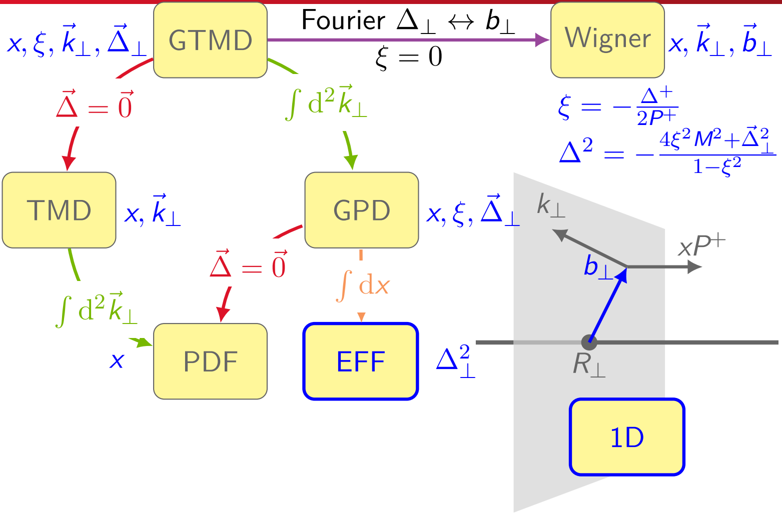
Light-cone physics

5-dimensional Wigner distribution

GPD properties



Back to Wigner distributions.



Back to Wigner distributions.

Exclusive reactions as a nuclear manometer
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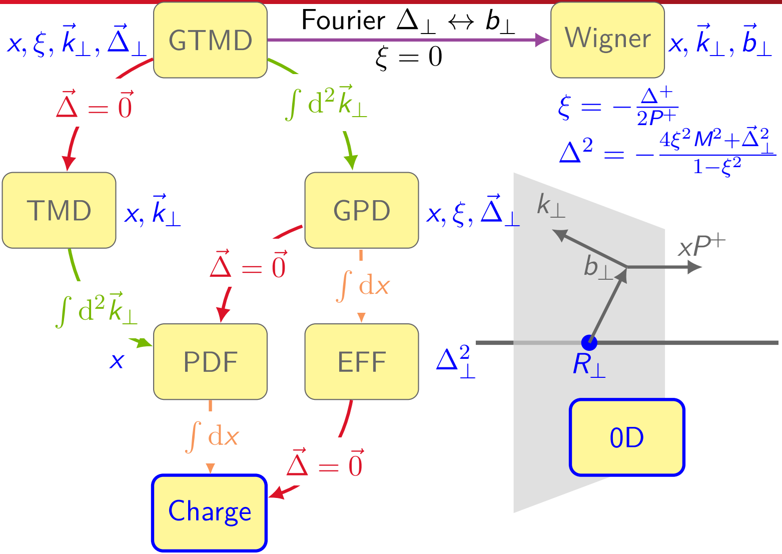

Back to Wigner distributions.

The family of 1-quark distributions.

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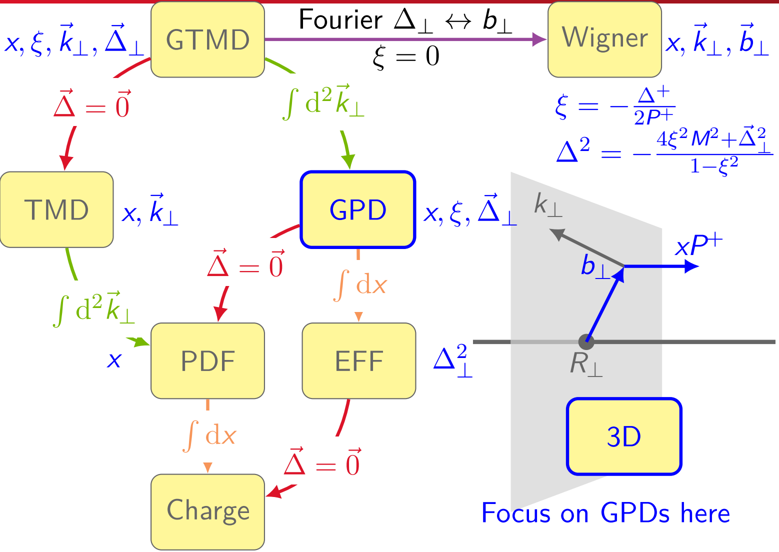

Back to Wigner distributions.

Polynomiality.

Mixed constraint from Lorentz invariance and discrete symmetries.

- Express Mellin moments of GPDs as **matrix elements**:

$$\int_{-1}^{+1} \mathrm{d}x \, x^m H^q(x, \xi, t) = \frac{1}{2(P^+)^{m+1}} \left\langle P + \frac{\Delta}{2} \middle| \bar{q}(0) \gamma^+ (i \overleftrightarrow{D}^+)^m q(0) \middle| P - \frac{\Delta}{2} \right\rangle$$

- Identify the **Lorentz structure** of the matrix element:

$$\text{linear combination of } (P^+)^{m+1-k} (\Delta^+)^k \text{ for } 0 \leq k \leq m+1$$

- Remember definition of **skewness** $\Delta^+ = -2\xi P^+$.
- Select **even powers** to implement time reversal.
- Obtain **polynomiality condition**:

$$\int_{-1}^{1} \mathrm{d}x \, x^m H^q(x, \xi, t) = \sum_{\substack{i=0 \\ \text{even}}}^{m} (2\xi)^i C^q_{mi}(t) + (2\xi)^{m+1} C^q_{mm+1}(t) \; .$$

- Choose $F^q(\beta, \alpha) = 3\beta\theta(\beta)$ ad $G^q(\beta, \alpha) = 3\alpha\theta(\beta)$:

$$H^q(x, \xi) = 3x \int_\Omega \mathrm{d}\beta \mathrm{d}\alpha \, \delta(x - \beta - \alpha\xi)$$

- Simple analytic expressions for the GPD:

$$H(x, \xi) = \frac{6x(1-x)}{1-\xi^2} \text{ if } 0 < |\xi| < x < 1,$$

$$H(x, \xi) = \frac{3x(x+|\xi|)}{|\xi|(1+|\xi|)} \text{ if } -|\xi| < x < |\xi| < 1.$$

Double Distributions.

Lorentz covariance by example.

- Compute first Mellin moments.

n	$\int_{-\xi}^{+\xi} \mathrm{d}x\, x^n H(x,\xi)$	$\int_{+\xi}^{+1} \mathrm{d}x\, x^n H(x,\xi)$	$\int_{-\xi}^{+1} \mathrm{d}x\, x^n H(x,\xi)$
0	$\frac{1+\xi-2\xi^2}{1+\xi}$	$\frac{2\xi^2}{1+\xi}$	1
1	$\frac{1+\xi+\xi^2-3\xi^3}{2(1+\xi)}$	$\frac{2\xi^3}{1+\xi}$	$\frac{1+\xi^2}{2}$
2	$\frac{3(1-\xi)(1+2\xi+3\xi^2+4\xi^3)}{10(1+\xi)}$	$\frac{6\xi^4}{5(1+\xi)}$	$\frac{3(1+\xi^2)}{10}$
3	$\frac{1+\xi+\xi^2+\xi^3+\xi^4-5\xi^5}{5(1+\xi)}$	$\frac{6\xi^5}{5(1+\xi)}$	$\frac{1+\xi^2+\xi^4}{5}$
4	$\frac{1+\xi+\xi^2+\xi^3+\xi^4+\xi^5-6\xi^6}{7(1+\xi)}$	$\frac{6\xi^6}{7(1+\xi)}$	$\frac{1+\xi^2+\xi^4}{7}$

- Expressions get more complicated as n increases... But they always yield polynomials!

Back to GPD properties.

Positivity.

A consequence of the positivity of the nom in a Hilbert space.

- Identify the matrix element defining a GPD as an **inner product** of two different states.
- Apply Cauchy-Schwartz inequality, and identify PDFs at specific kinematic points, *e.g.*:

$$|H^q(x,\xi,t)| \leq \sqrt{\frac{1}{1-\xi^2} q\left(\frac{x+\xi}{1+\xi}\right) q\left(\frac{x-\xi}{1-\xi}\right)}$$

- This procedures yields **infinitely many inequalities** stable under LO evolution.

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- The **overlap representation** guarantees *a priori* the fulfillment of positivity constraints.

Back to GPD properties.

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