Nuclear equation of state from ground and excited state properties of nuclei

1.-Introduction and phenomenology

Xavier Roca-Maza

Università degli Studi di Milano

INFN sezione di Milano

École Joliot-Curie

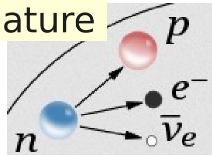
“Nuclear Matter under Pressure”

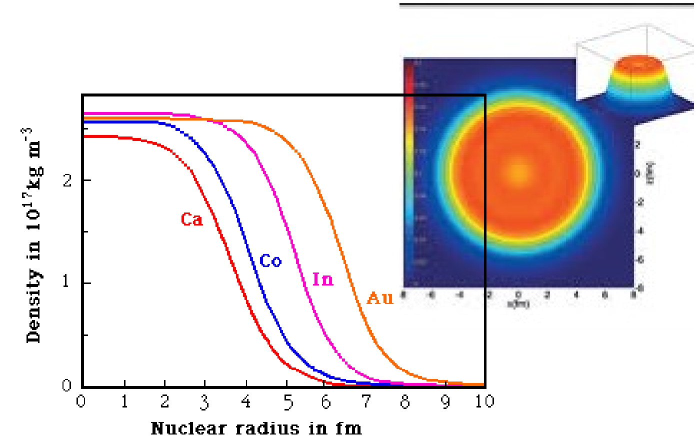
Saint Pierre d’Oleron,

France September 4th-9th

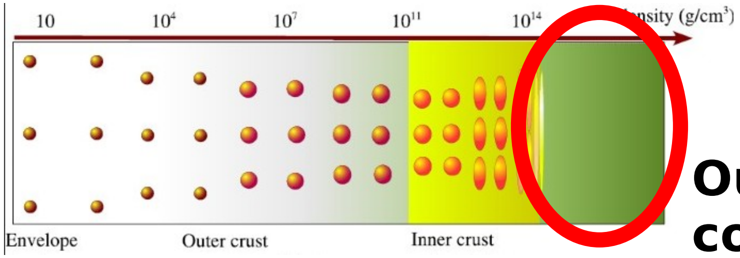
Where can we find neutrons and protons? And in which form? Free? In clusters?

- Neutrons and protons in **Earth** are found in cluster systems: **nuclei**
  - → The **interior** of all nuclei has **constant density** (10^{14} times denser than water) named **saturation density**
  - → Saturation is originated from the **short range** nature of the **nuclear effective interaction**
  - → Neutron in 15 minutes must find a proton or …

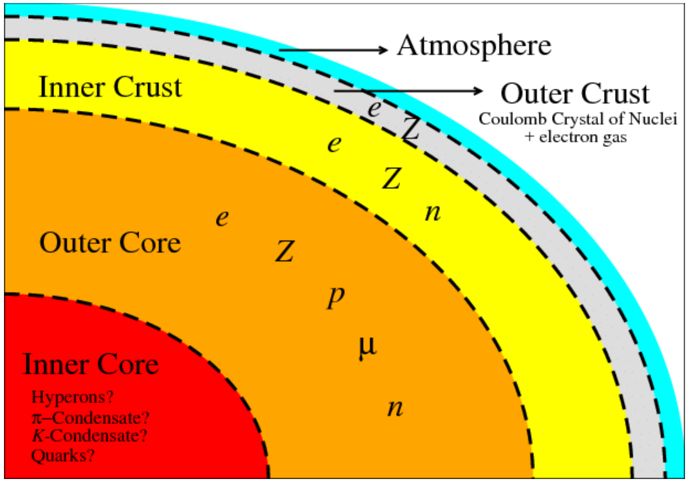




- In **heavens**, neutrons and protons can be also found as an interacting sea of fermions (Fermi liquid): **matter in the outer core of a neutron star**
  - → Densities can reach several times nuclear saturation



Outer core



Nuclear Equation of State (EoS)

Definition: the **energy** per **nucleon** ($e=E/A$ where $A=N+Z$) of an **uniform system of neutrons and protons** as a function of the **neutron** (ρ_n = N/V) and **proton** (ρ_p = Z/V) **densities**, at **zero temperature**, **unpolarized**, assuming **isospin symmetry** and **neglecting Coulomb** effects among protons.

Why???

→ **Zero temperature:** room temperature 10^2K→$\mathbf{10^{-8}}$ **MeV** while "cold" neutron stars are at about 10^{10}K→1 MeV. **Separation energy** in stable nuclei (equivalent to ionization energy in atoms) is of **several MeV**.

→ **Unpolarized:** energy favours **couples** of neutrons and protons **occupying the same state but with opposite spins** (equivalent to electrons in atoms)

→ **Isospin symmetry:** neutron-neutron, proton-proton and neutron-proton **nuclear interaction** are very **similar** among them. **Masses** of neutrons and protons are almost **degenerate**. Hence neutrons and protons can be thought as **two states** of the **same particle** with different isobaric spin or **isospin** (in analogy with spin): the **nucleon.**

→ **No Coulomb: idealized uniform system (focus on strong interaction)**. Real systems are finite and frequently electrically neutral so no problems (divergences) in adding Coulomb.

[Besides that, the strong interaction at the typical scale of a nucleus is much stronger than the Coulomb interaction and the Coulomb energy (*) per particle of an infinite system of protons would be infinite.]

$$(*)\text{Coulomb interaction infinite range!!} \rightarrow \frac{E_C}{A} \sim \frac{Z(Z-1)}{A^{4/3}} \sim Z^{2/3}$$

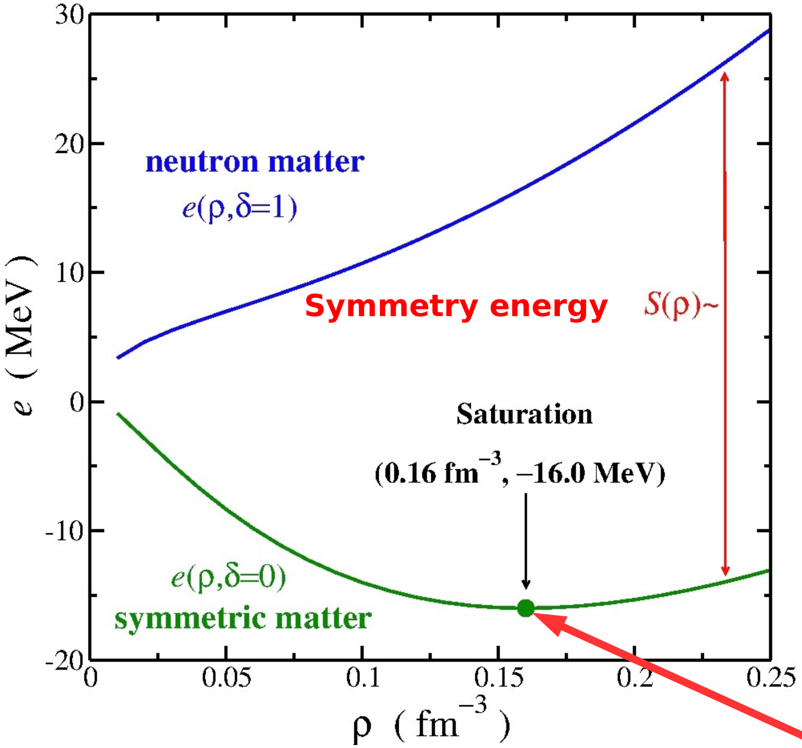
Nuclear Equation of State (EoS)

It is **convenient** to write the **energy per nucleon** (e) as a **function** of the **total density** [$\rho=\rho_n+\rho_p$] and their **relative difference** [$\delta=(\rho_n+\rho_p)/\rho$].

→ Due to isospin symmetry only even powers of δ will appear

→ Stable nuclei tend to show small values of δ

Taylor expansion for δ→ 0: $$e(\rho,\delta) = e(\rho,0) + S(\rho)\delta^2 + \mathcal{O}[\delta^4]$$



It is customary to also **expand** e(ρ,0) and S(ρ) around nuclear **saturation density** $\rho_0 \sim 0.16$ fm^{-3}

$$e(\rho,0) = e(\rho_0,0) + \frac{1}{2}K_0 x^2 + \mathcal{O}[\rho^3] \text{ where } x = \frac{\rho-\rho_0}{3\rho_0}$$

$$S(\rho) = J + Lx + \frac{1}{2}K_{\text{sym}}x^2 + \mathcal{O}[\rho^3,\delta^4]$$

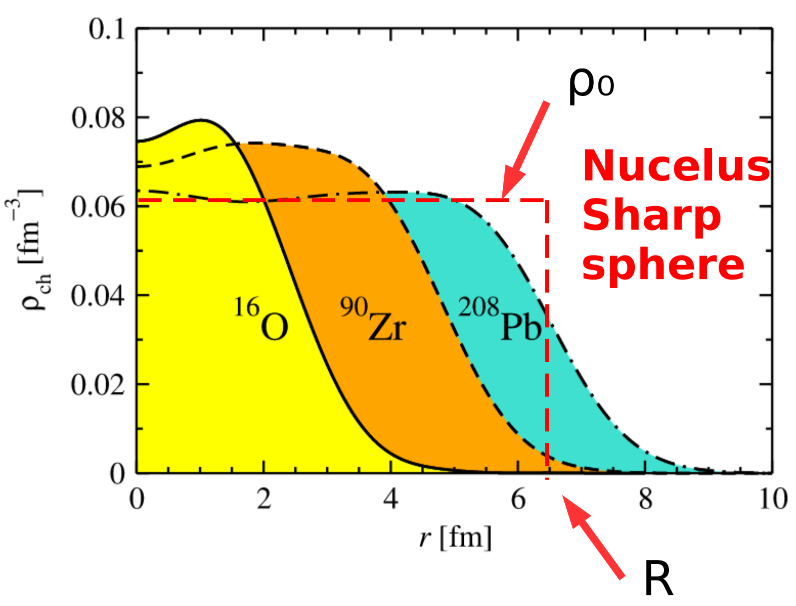
K_0 → how **compressible** is symmetric matter at ρ_0

J (a_A) → **penalty energy** for converting all **protons into neutrons** in symmetric matter at ρ_0

L (a_s) → **neutron pressure** in neutron matter at ρ_0

$$P(\rho=\rho_0,\delta=0) = 0 \text{ MeV fm}^{-3}$$

Saturation density $\rho_0 \approx 0.16\ fm^{-3}$



→ **Range of the nuclear interaction** (1/m_π ~1-2 fm) typically **shorter** than the **size** of the **nucleus.** Hence, neutrons and protons just "see" their closest neighbours.

→ **Experimental charge (Z) density** in the interior of very **different nuclei** is rather constant at around **0.06-0.08 fm^{-3}**.

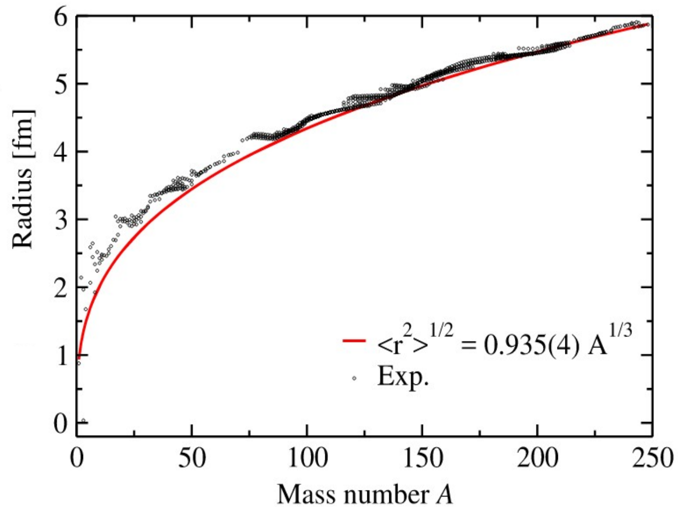
→ **Saturation mechanism** (equilibrium) that originates from the **short-range nature of the nuclear force,** much stronger than the Coulomb repulsion at the nuclear scale.

$$N + Z \equiv A = \int d\mathbf{r}\rho(\mathbf{r}) \xrightarrow{\text{sharp sphere}} A = \frac{4}{3}\pi\rho_0 R^3 \rightarrow R = \left(\frac{3}{4\pi\rho_0}\right)^{1/3} A^{1/3}$$

$$\langle r^2 \rangle^{1/2} \equiv \left(\frac{1}{A}\int d^3r\, r^2 \rho\right)^{1/2} \approx \sqrt{\frac{3}{5}}R = \sqrt{\frac{3}{5}}\left(\frac{3}{4\pi\rho_0}\right)^{1/3} A^{1/3}$$

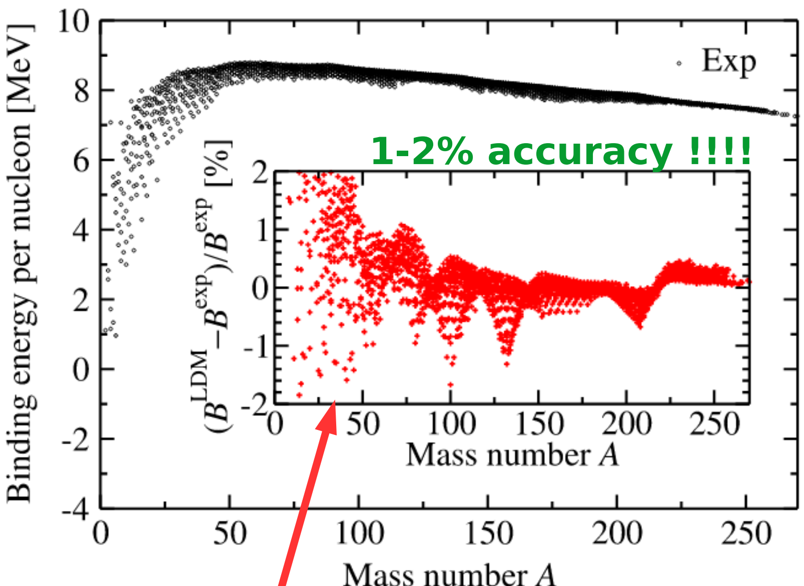
$$R \approx r_0 A^{1/3}$$

≈0.9 fm

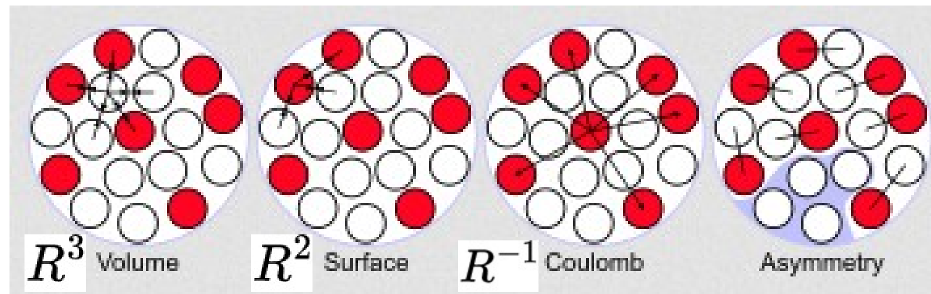


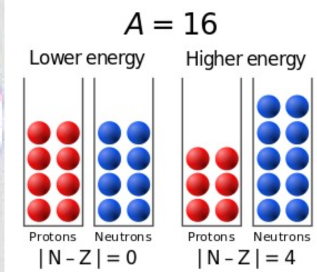
Energy at saturation density:

energy of a nucleon “far from the surface” → $a_V \approx 16$ MeV



→ **Nucleus seen as an incompressible liquid (ideal) drop: sharp sphere of radius $R \approx r_0 A^{1/3}$**





$$M(A,Z) = m_p Z + m_n(A-Z) - B(A,Z)$$

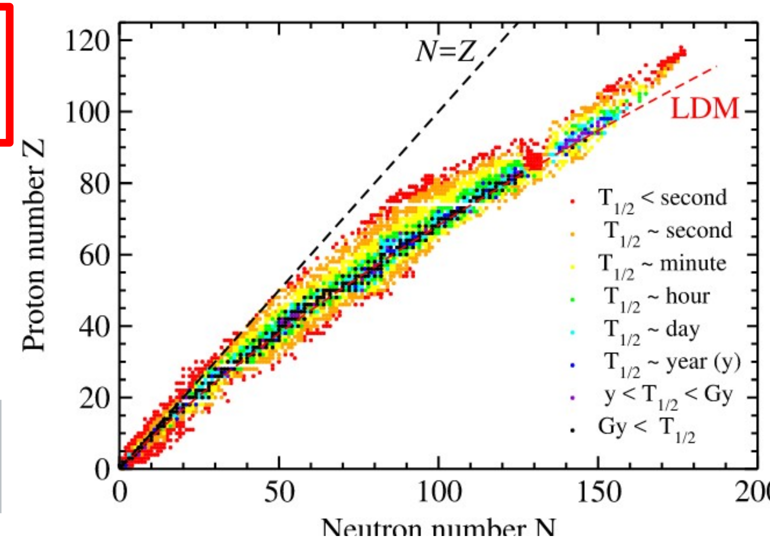
$$B(A,Z) = (a_V - a_S A^{-1/3})A - a_C \frac{Z(Z-1)}{A^{1/3}} - (a_A - a_{AS} A^{-1/3}) \frac{(A-2Z)^2}{A}$$

Arch structure in the **residuals** ↔ **shell structure** not acounted by the model, **effects** about **few %**

Stability of M(A,Z) with respect to **Z** →

$$\frac{Z}{A} \approx \frac{1}{2} \frac{1}{1 + \frac{a_C}{4(a_A - a_{AS} A^{-1/3})} A^{2/3}}$$

Competition between Coulomb (Z→0) and asymmetry (N→Z) and surface term



Important!!

→ A **small change** in the **saturation density** will **impact** the **size** of the **nucleus.** **Charge radii** are determined to an average accuracy of 0.016 fm (Angeli 2013).

For example, if one aims at determining the $\mathbf{r_{ch}}$ **= 5.5012±0.0013 fm** in ^{208}**Pb** one must be **very precise** in the determination of $\boldsymbol{\rho_0}$:

$$\frac{\delta\rho_0}{\rho_0} = -3\frac{\delta R}{R} \rightarrow \frac{\delta\rho_0}{\rho_0} \lesssim 0.1\%$$

Note: typical average theoretical deviation of accurate nuclear models ~ 0.02 fm → $\boldsymbol{\delta\rho_0/\rho_0}$ is determined up to about a **1% accuracy** (That is, third digit in $\rho_0 \approx 0.16$ fm^{-3}**!!**).

→ In a similar way, a **small change** in the **saturation energy** (about $e_0 \approx -16$ MeV) will **impact** on the **nuclear mass.**

For example, if one aims at determining the **B = 1636.4296±0.0012 MeV** in ^{208}**Pb** one must be **very precise** in the determination of $\mathbf{e_0}$ (changed notation!):

$$\frac{\delta B}{B} = \frac{\delta e_0}{e_0} \rightarrow \frac{\delta e_0}{e_0} \lesssim 10^{-6}$$

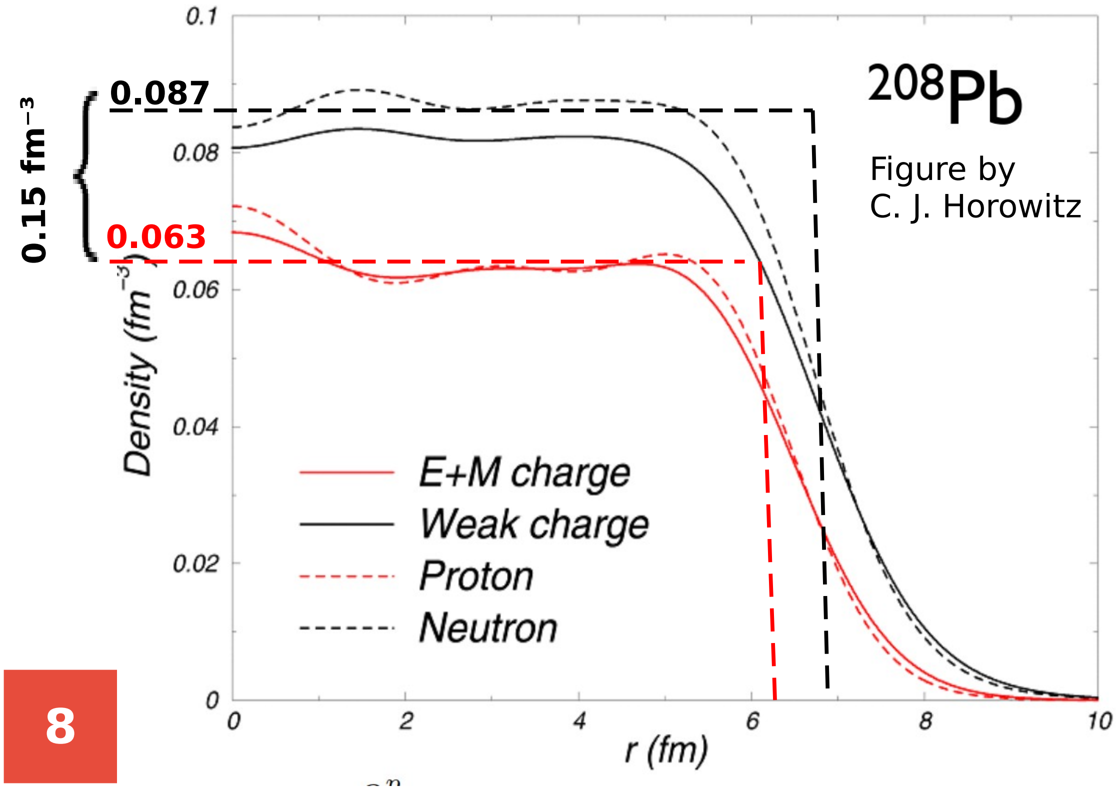
Note: typical average theoretical deviation of accurate nuclear models ~ 1-2 MeV → $\boldsymbol{\delta e_0/\rho e_0}$ is determined up to about a **0.1% accuracy** (That is, second decimal digit in $e_0 \approx -16.0$ MeV**!!**).

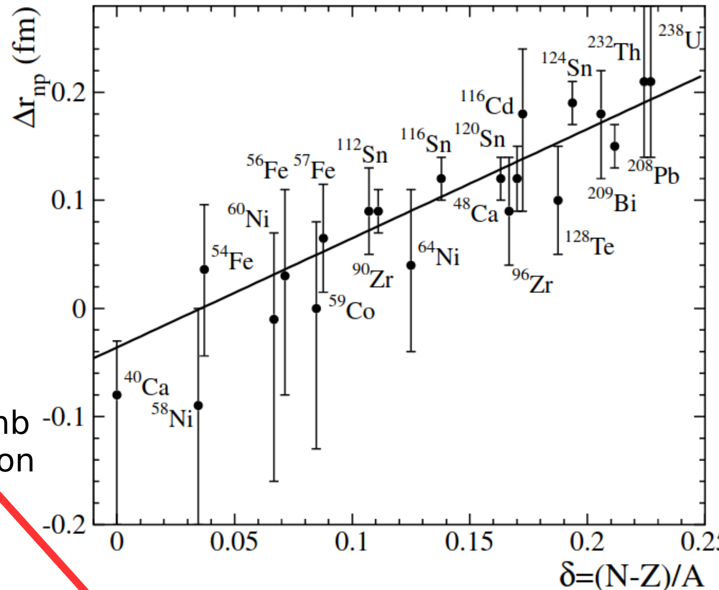
Neutron and proton radii difference

essentially due to the difference between N and Z $\Delta r_{np} \equiv \langle r_n^2 \rangle^{1/2} - \langle r_p^2 \rangle^{1/2}$

- Elastic electron scattering → electromagnetic size of the nucleus ↔ $\boldsymbol{\rho_p}$
- We have mostly indirect measurements on $\boldsymbol{\rho_n}$ (weakly interacting probes difficult)
- In nuclei with **different** number of **neutrons** and **protons**, we expect $\mathbf{R_n}$ could be different from $\mathbf{R_p}$:

$$\frac{\langle r_n^2 \rangle^{1/2} - \langle r_p^2 \rangle^{1/2}}{\langle r^2 \rangle^{1/2}} \approx \frac{N^{1/3} - Z^{1/3}}{A^{1/3}} \xrightarrow[I \ll 1]{I \equiv (N-Z)/A} \frac{\langle r_n^2 \rangle^{1/2} - \langle r_p^2 \rangle^{1/2}}{\langle r^2 \rangle^{1/2}} \propto \frac{N-Z}{A}$$





Coulomb repulsion → R_p larger

$$\Delta r_{np} = -0.04(3) + 1.01(15)\frac{N-Z}{A} \text{ fm}$$

A. Trzcińska, J. Jastrzębski, P. Lubiński, F. J. Hartmann, R. Schmidt, T. von Egidy, and B. Kłos
Phys. Rev. Lett. **87**, 082501 – Published 2 August 2001

Neutron skin thickness ($\Delta r_{np} := r_n - r_p$) and neutron pressure

For a fixed **(N-Z)/A**, one must **expect** that the **larger the pressure felt by nucleons**, **the larger the skin**

$$P = -\frac{\partial E}{\partial V}\bigg|_A = \rho^2 \frac{\partial e(\rho,\delta)}{\partial \rho}\bigg|_\delta =$$

$$\rho^2 \frac{\partial}{\partial \rho}\left[e(\rho,0) + S(\rho)\delta^2\right] =$$

$$\rho^2\delta^2 \frac{\partial S(\rho)}{\partial \rho} = \frac{1}{3}\rho\delta^2 L$$

→ **From the Droplet Model:**

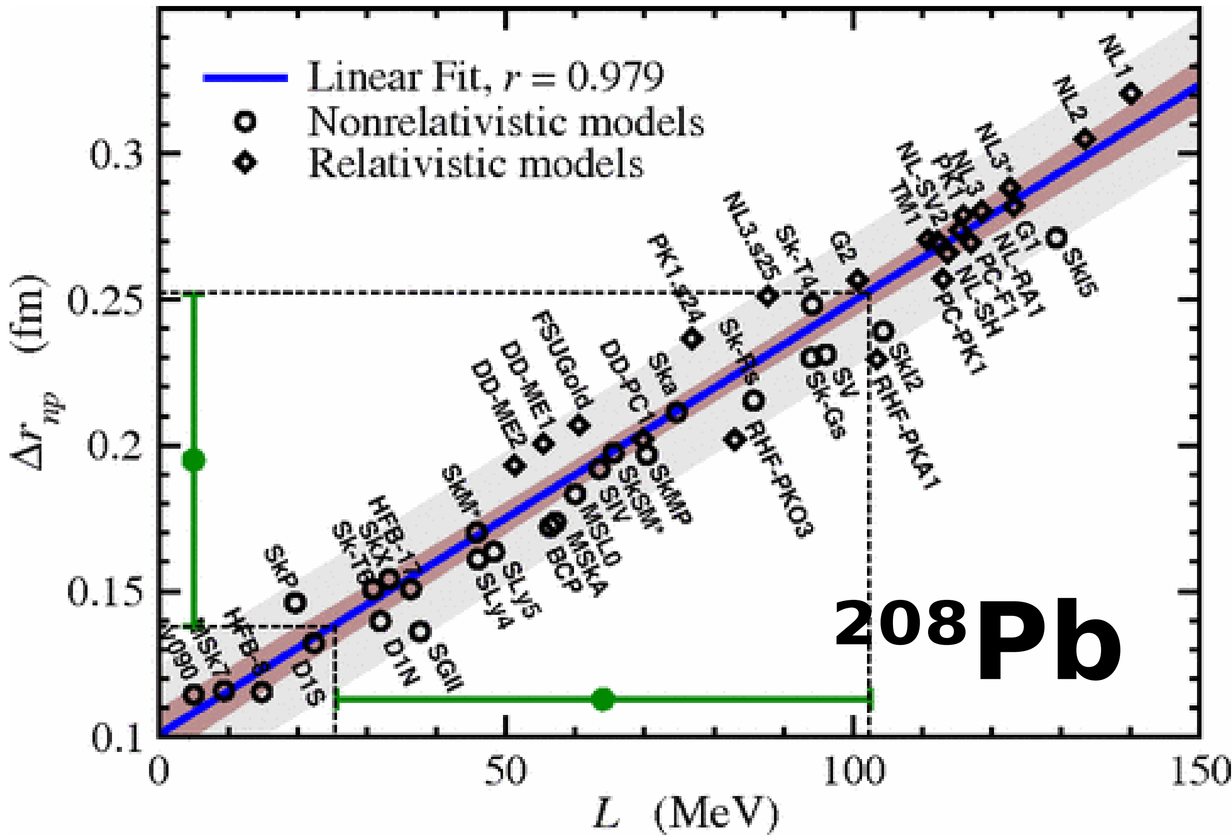
$$\Delta r_{np} \approx \frac{1}{12}\frac{N-Z}{A}\frac{R}{J}L$$

The nuclear droplet model for arbitrary shapes

W.D Myers, W.J Swiate

Annals of Physics

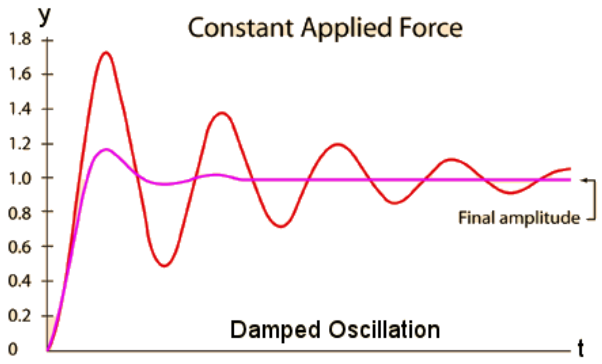
Volume 84, Issues 1–2, 15 May 1974, Pages 186-210



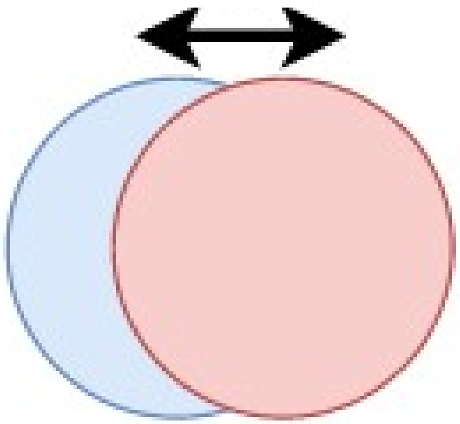
Neutron Skin of 208Pb, Nuclear Symmetry Energy, and the Parity Radius Experiment

X. Roca-Maza, M. Centelles, X. Viñas, and M. Warda Phys. Rev. Lett. 106, 252501 (2011)

What happens if we now perturb the ground state densities?



Produce a **small displacement (dl)** between **neutron and proton densities** (drops)



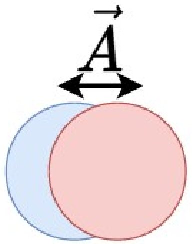
$$\rho = \rho_0 + \delta\rho_0 \approx$$

$$\rho_0 + d\vec{l} \cdot \vec{\nabla}\rho_0$$

(Linear response theory)

Under different types of perturbations, **nuclei use to show ressonant behaviours** where all nucleons oscillate coherently and the nucleus as a whole vibrate at an specific resonant energy → known as **Giant Resonances**

Giant resonances: the IVGDR



→ The **Isovector Giant Dipole Resonance** was the first resonance measured (**photo-absorption** experiments)

→ The **cross section** for the **excitation** of the nucleus to a **final state** |ν⟩ with energy $\mathbf{E_\nu}$ from the **ground state** |0⟩ with energy $\mathbf{E_0}$ by a **photon** at a given energy **E** can be written as

$$\sigma_\nu(E) = 4\pi^2\alpha(E_\nu - E_0)|\langle\nu|F_{\text{dipole}}|0\rangle|^2\delta(E - E_\nu + E_0)$$

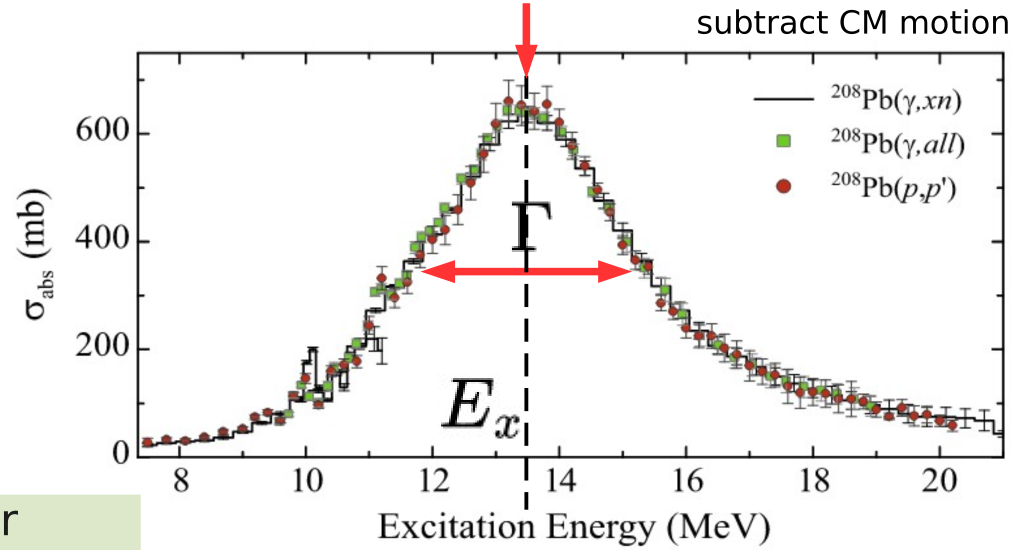
Convenient operator [~ r $Y_{10}(\Omega)$] produces dipole transitions and subtract CM motion

→ The **total cross-section** will be

$$\sigma_{\gamma-\text{abs}} = 4\pi^2\alpha\sum_\nu(E_\nu - E_0)|\langle\nu|F_{\text{dipole}}|0\rangle|^2$$

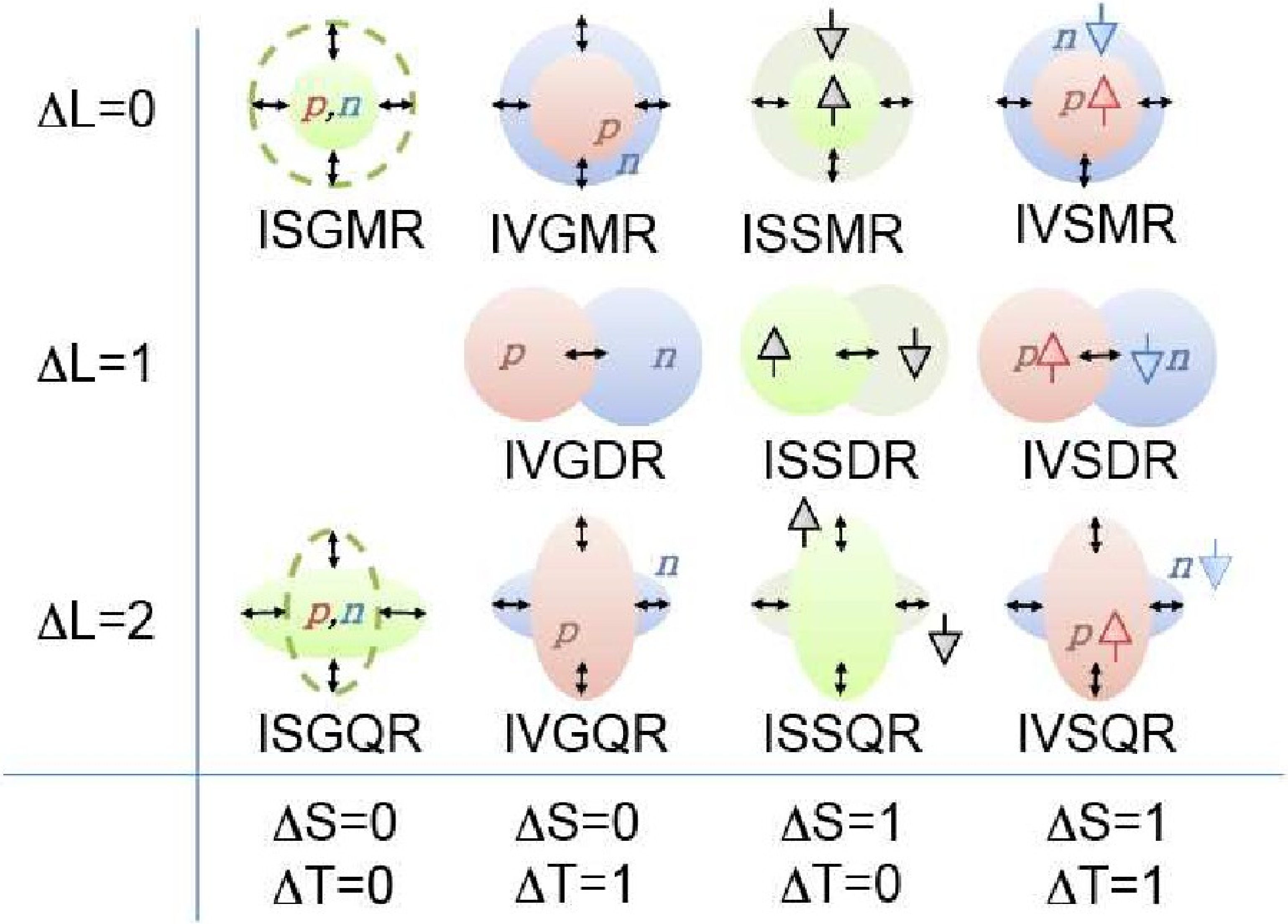
$$S(E) \equiv \sum_\nu |\langle\nu|F|0\rangle|^2\delta(E - E_\nu + E_0)$$

where **S(E)** is the so called **Strength function**



S(E) is used to **characterize** the nuclear **response** (experimentally, it is commonly parametrized as a Lorentzian function with energy E_x and width Γ)

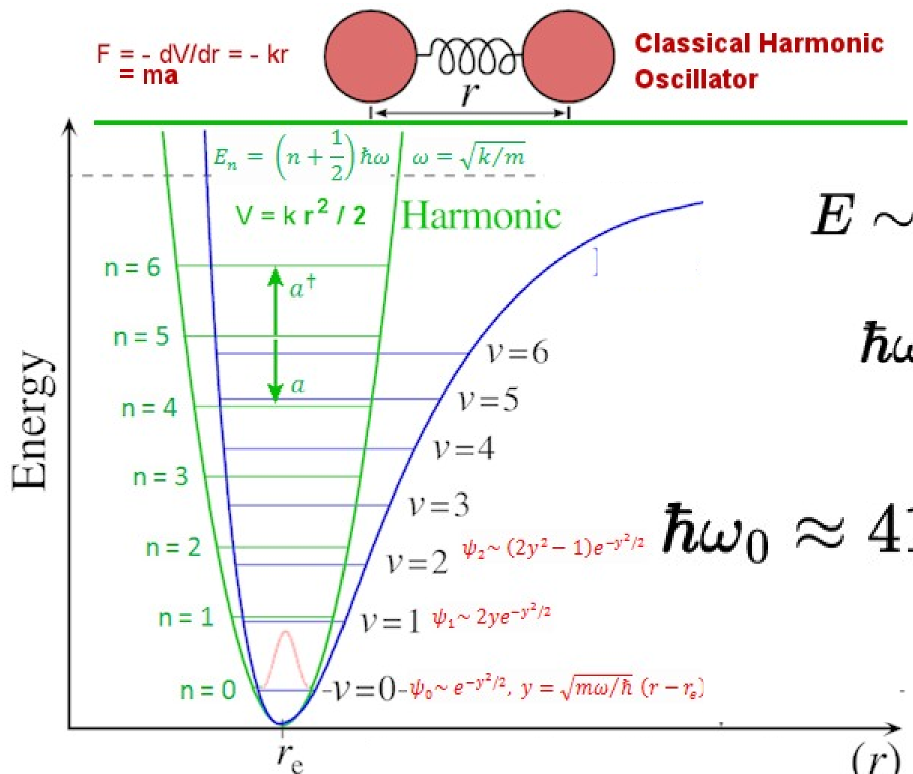
Giant Resonances



http://www.majimak.com/wordpress/

Giant Resonances: Harmonic oscillator

Assuming **nucleons** as **non-interacting** fermions confined in an **Harmonic Oscillator** (HO) trap with suitable **hω₀** that **preserves** the main features of the g.s. **energy spectra**

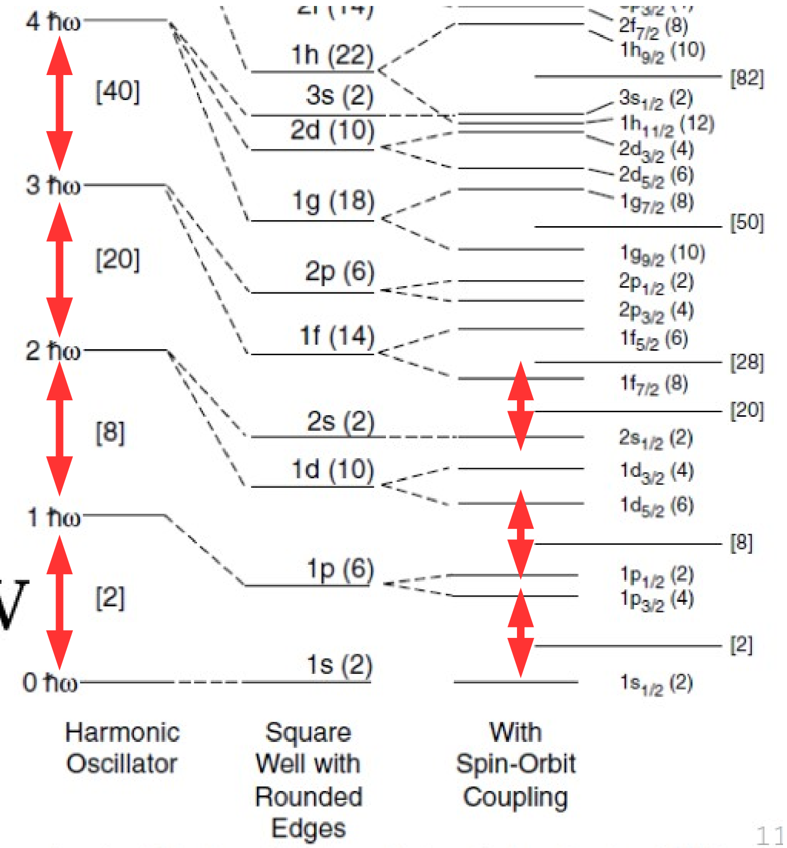


Shell gaps equal in HO

$$E \sim \frac{1}{R} \sim A^{-1/3}$$

$$\hbar\omega_0 \sim A^{-1/3}$$

$$\hbar\omega_0 \approx 41 A^{-1/3}\ \mathrm{MeV}$$



(modified from https://universe-review.ca/R15-33-harmonics.htm)

Giant Resonances: Harmonic oscillator

The **HO Hamiltonian**, in terms of the conjugate variables α (or r) and dα/dt (or v) and the ($C_0 \leftrightarrow m\omega^2$) and ($B_0 \leftrightarrow m$) parameters, could be written as (Bohr&Mottelson):

$$\mathcal{H}_0 = \frac{1}{2}B_0\dot{\alpha}^2 + \frac{1}{2}C_0\alpha^2 \rightarrow E_0 = \hbar\left(\frac{C_0}{B_0}\right)^{1/2}$$

$$E = \left(\frac{1}{B_0}\frac{\partial^2 U}{\partial \alpha^2}\right)^{1/2}$$

Assume harmonic perturbation (restoring force)

$$F = -\kappa\alpha \rightarrow V = \frac{1}{2}\kappa\alpha^2$$

$$\mathcal{H} = \mathcal{H}_0 + V \rightarrow \mathcal{H} = \frac{1}{2}B_0\dot{\alpha}^2 + \frac{1}{2}(C_0 + \kappa)\alpha^2$$

$$E = E_0\left(1 + \frac{\kappa}{C_0}\right)^{1/2}$$

Restoring force

Depending on the type of perturbation

$$E \propto \left(\frac{\partial^2 e}{\partial \delta^2}\right)^{1/2} = [S(\rho)]^{1/2}$$

$$E \propto \left(\frac{\partial^2 e}{\partial \rho^2}\right)^{1/2} = [K]^{1/2}$$

Sum rules:
Ground state gives access to excited state properties!!

→ **Sum Rules** or **moments** of the **strength function S(E):**

$$S(E) = \sum_\nu |\langle \nu || \hat{F}_J || \tilde{0} \rangle|^2 \delta(E - E_\nu + E_0)$$

k-moment of S(E): $$m_k = \int dE\, E^k S(E) = \sum_\nu |\langle \nu || \hat{F}_J || \tilde{0} \rangle|^2 (E_\nu - E_0)^k$$

Example: Energy Weighted Sum Rule (EWSR)

$$m_1 = \sum_\nu (E_\nu - E_0) |\langle \nu | F | 0 \rangle|^2 = \langle 0 | F^\dagger [\mathcal{H}, F] | 0 \rangle$$

Written in terms of a **commutator** with the **Hamiltonian** evaluated in the **G.S. !!**

→ [F,V]=0, if the excitation **operator commutes** with the **interaction** the **sum rule** will be **model independent!!**

$$\hat{F}_J^{(IS)} = \sum_{i=1}^{A} f_J(r_i) Y_{JM}(\hat{r}_i) \qquad m_1^{(IS)}(J) = \frac{\hbar^2}{2m} \frac{A}{4\pi} (2J+1) \langle g_J \rangle,$$

$$\langle g_J \rangle = \frac{1}{A} \int g_J(r) \rho(r) d^3 r,$$

$$g_J(r) = \left(\frac{df_J}{dr} \right)^2 + J(J+1) \left(\frac{f_J}{r} \right)^2$$

→ [F,V] different form 0, if the excitation **operator does not commute** with the **interaction** the **sum rule** will be **model dependent** but still can be **used** to better **understand** nuclear **phenomenology**

$$\hat{F}_J^{(IV)} = \sum_{i=1}^{A} f_J(r_i) Y_{JM}(\hat{r}_i) \tau_z(i) \qquad m_1^{(IV)}(J) = m_1^{(IS)}(J)(1 + \tilde{\kappa})$$

Model dependent term

Dielectric theorem:

Inverse Energy Weighted Sum Rule m_{-1}

Ground state |0> **perturbed** by an **external field** λF (λ → 0) so that perturbation theory holds → The **expectation value** of the **Hamiltonian** <H> and of the **operator** <F> can be written:

$$\delta\langle\mathcal{H}\rangle = \lambda^2 \sum_{\nu\neq 0} \frac{|\langle\nu|F|0\rangle|^2}{E_\nu - E_0} + \mathcal{O}(\lambda^3) = \lambda^2 m_{-1} + \mathcal{O}(\lambda^3)$$

$$\delta\langle F\rangle = -2\lambda \sum_{\nu\neq 0} \frac{|\langle\nu|F|0\rangle|^2}{E_\nu - E_0} + \mathcal{O}(\lambda^2) = -2\lambda m_{-1} + \mathcal{O}(\lambda^2)$$

$$m_{-1} = \frac{1}{2}\frac{\partial^2\langle\mathcal{H}\rangle}{\partial\lambda^2}\bigg|_{\lambda=0} = -\frac{1}{2}\frac{\partial\langle F\rangle}{\partial\lambda}\bigg|_{\lambda=0} \quad \rightarrow \quad \frac{1}{m_{-1}} = 2\frac{\partial^2\langle\mathcal{H}\rangle}{\partial\langle F\rangle^2}$$

Use of sum rules:
Giant Monopole Resonance

→ Is the **nucleus compressible** or it is as in the **Liquid Drop Model?** (an ideal incompressible liquid)

The thermodynamic definiton of compressibility is: $\chi = \frac{1}{V}\left(\frac{\partial P}{\partial V}\right)^{-1}$

The K_0 parameter (slide 4) can be easily related to χ from its definition

$$\chi = -\frac{1}{V}\left[\frac{\partial}{\partial V}\left(\frac{\partial E}{\partial V}\right)\right]^{-1}_{A=\text{cons.}} = \frac{9}{\rho K_0}$$

So far this is for the uniform system, what about the nucleus?

$$\chi = \frac{1}{V}\left(\frac{\partial P}{\partial V}\right)^{-1} \xrightarrow{\text{Spherical symmetry}} \frac{1}{\chi} = \frac{r}{3}\left(-rP + \frac{1}{4\pi r^2}\frac{\partial^2 E}{\partial r^2}\right)$$

Nucleus at equillibrium → P = 0. In analogy, we can define $K_A \equiv 9V/\chi$

$$K_A = \frac{9V}{\chi} = 9\frac{r^2}{9}\frac{\partial^2 E}{\partial r^2} = Ar^2\frac{\partial^2 (E/A)}{\partial r^2} = 4A(r^2)^2\frac{\partial^2 E/A}{\partial (r^2)^2}$$

Now, from the moments of S(E), one can define an excitation energy

$$E_x^{\text{centroid}} = \frac{\int ES(E)dE}{\int S(E)dE}; \quad E_x^{\text{constrained}} = \sqrt{\frac{\int ES(E)dE}{\int S(E)/E\, dE}}; \quad E_x^{\text{scaling}} = \sqrt{\frac{\int E^3 S(E)dE}{\int ES(E)dE}}$$

Use of sum rules:
Giant Monopole Resonance

In our case, we will use the **constrained energy** since it is easy to calculate.

The operator leading to monopole transitions (isotropic changes in the volume if we think about a liquid drop) cannot depend on the orbital angular momentum or spin:

$$F = \sum_{i=1}^{A} r_i^2 \quad (*)$$

Isotropic harmonic perturbation!

The m_1 and m_{-1} moments are:

$$m_1 = \frac{2\hbar^2}{m}\langle r^2\rangle \qquad \frac{1}{m_{-1}} = 2\frac{\partial^2\langle\mathcal{H}\rangle}{\partial\langle r^2\rangle^2}$$

Therefore,

$$(E_x^{\mathrm{ISGMR}})^2 = \frac{m_1}{m_{-1}} = 4\frac{\hbar^2}{m}\langle r^2\rangle\frac{\partial^2 E}{\partial\langle r^2\rangle^2} = 4A\frac{\hbar^2}{m\langle r^2\rangle}\langle r^2\rangle^2\frac{\partial^2(E/A)}{\partial\langle r^2\rangle^2} \equiv K_A\frac{\hbar^2}{m\langle r^2\rangle}$$

Ok, we have now defined the **incompressibilty** of a finite nucleus and **connected** it to an **experimentally measurable quantity**. Can we say something about the EoS?

(*) other definitions could differ by a factor 4π depending if $Y_{00} = 1/\mathrm{sqrt}(4\pi)$ included or not

Use of sum rules:
Giant Monopole Resonance

Nuclear compressibilities

J.P. Blaizot

Physics Reports

Volume 64, Issue 4, September 1980, Pages 171-248

Assuming a **Liquid Drop Model** like expansion for $\mathbf{K_A}$ one can connect it to the bulk incompressibility $\mathbf{K_0}$ (also named “leptodermus” expansion) of the **nuclear EoS**

$$K_A = K_0 + K_s A^{-1/3} + K_\tau \left(\frac{N-Z}{A}\right)^2 + K_C \frac{Z(Z-1)}{A^{4/3}} + \ldots$$

Fitting to the excitation energy of the ISGMR one would obtain the coefficients of this formula. Among them K_0 (recent estimated accuracy over 10% Phys. Rev. C **89**, 044316)

This formula is **qualitative** since misses **shell effects** and **pairing** as well as terms in the **expansion** that goes as powers of **A** and **(N-Z)/A**. Very much like the LDM. Hence the estimation of K_0 would have large systematic (theoretical) errors

For the description of ^{208}Pb (E_x=13.6±0.5 MeV), $\mathbf{K_0}$ must be determined at about **7%** accuracy or better

$$\left(\frac{\delta K_0}{K_0}\right)^2 = \left(2\frac{\delta E_x}{E_x}\right)^2 + \left(2\frac{\delta\langle r^2\rangle^{1/2}}{\langle r^2\rangle^{1/2}}\right)^2 \approx \left(2\frac{\delta E_x}{E_x}\right)^2$$

$$\frac{\delta K_0}{K_0} \approx 2\frac{\delta E_x}{E_x} \approx 7\%$$

Use of sum rules:
Giant Monopole Resonance

Looking at the monolope from a different perspective:

→ For an isoscalar and velocity independent operator like the monopole operator:
$$m_3(F) = \sum_\nu (E_\nu - E_0)^3 |\langle \nu | F | 0 \rangle|^2 = \langle 0 | F^\dagger [\mathcal{H}, [\mathcal{H}, [\mathcal{H}, F]]] | 0 \rangle$$
$$\equiv \langle 0 | \tilde{F} [\mathcal{H}, \tilde{F}] | 0 \rangle = m_1(\tilde{F}) \qquad \text{for} \quad \tilde{F} = i[H, F]$$

→ Assuming an scaling of Ψ as: $\Psi_\eta = \exp\left(-i\eta\tilde{F}\right)\Psi$

One finds [PHYSICS REPORTS 51, No. 5(1979) 267-316] $m_3(F) = \frac{1}{2} \frac{\partial^2 \langle \mathcal{H} \rangle}{\partial \eta^2}\Big|_{\eta=0}$

→ Allows to calculate the scaling energy ($E^2 = m_3/m_1$) in microscopic Hamiltonians solved, e.g, within the Hartree-Fock approximation using the monopole operator (r^2) as:

$$\mathcal{H} = \sum_i \frac{\hbar^2 \nabla_i^2}{2m} + \frac{1}{2} \sum_{i \neq j} V(r_i, r_j)$$
$$V(r_i, r_j) = A_{\text{contact}} \delta(r_i - r_j) + V_C + V_{\text{SO}} + \ldots$$

→

$$(E_x^{\text{ISGMR}})^2 = \frac{\hbar^2}{mA\langle r^2 \rangle} [4T + 9E_{\text{contact}} + 25E_{\text{SO}} + \ldots]$$

Use of sum rules:
Giant Monopole, Quadrupole, etc,... Resonances

Now, **combining** this result with the expression within the same model for the **binding energy**, the excitation energy of the **quadrupole GR** [$F=r^2\, Y_{20}(\Omega)$], etc ... one could build a **system** of equations to be **solved** given available **experimental data**

$$E = T + E_{\mathrm{contant}} + E_{\mathrm{SO}} + E_C + \ldots$$

$$(E_x^{\mathrm{ISGMR}})^2 = \frac{\hbar^2}{mA\langle r^2\rangle}[4T + 9E_{\mathrm{contact}} + 25E_{\mathrm{SO}} + \ldots]$$

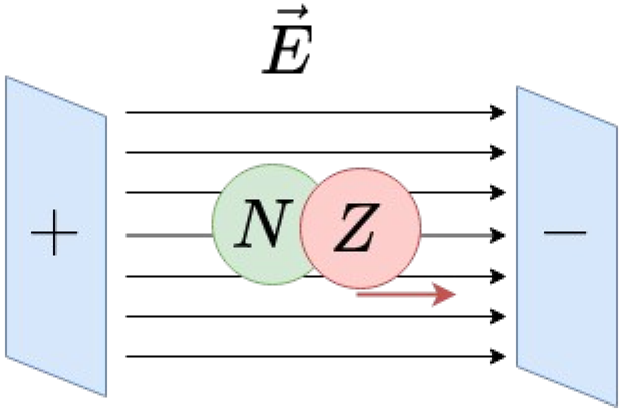
$$(E_x^{\mathrm{ISGQR}})^2 = \frac{4\hbar^2}{mA\langle r^2\rangle}[T + \frac{1}{4}E_{\mathrm{SO}} - \frac{1}{5}E_C + \ldots]$$

$$\ldots$$

Use of sum rules:
Dipole polarizability (Giant Dipole Resonance)

As in Electromagnetism course in the Physics degree, the **electric polarizability** measures **tendency** of the nuclear **charge distribution** to be **distorted**

$$\alpha = \frac{\text{electric dipole moment}}{\text{external electric field applied}}$$

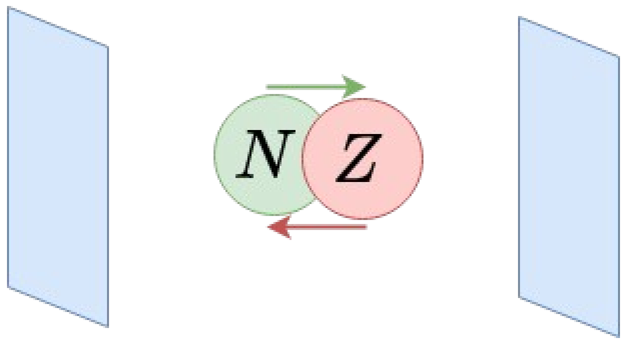


Polarizability is **proportional** to the inverse energy weighted sum rule $\mathbf{m_{-1} = \Sigma\, S(E)/E}$ (response function theory)

How easy is to separate neutrons from protons? Symmetry energy will tell. Remember the HO model?

$$e(\rho,\delta) = e(\rho,0) + S(\rho)\delta^2$$

$$E_x \sim \sqrt{\frac{\partial^2 e(\rho,\delta)}{\partial \delta^2}} \sim \sqrt{S(\rho)}$$



Use of sum rules:
Dipole polarizability (Giant Dipole Resonance)

→ Calculate the polarizzability (α), **proportional to m-1** from the **dielectric theorem** and **Droplet Model** ($J=a_A$)

$$\alpha_D = \frac{8\pi e^2}{9} m_{-1}(E1)$$

$$m_{-1} \approx \frac{A\langle r^2\rangle^{1/2}}{48J}\left(1+\frac{15}{4}\frac{J}{Q}A^{-1/3}\right)$$

J. Meyer, P. Quentin, and B. Jennings, Nucl. Phys. A **385**, 269

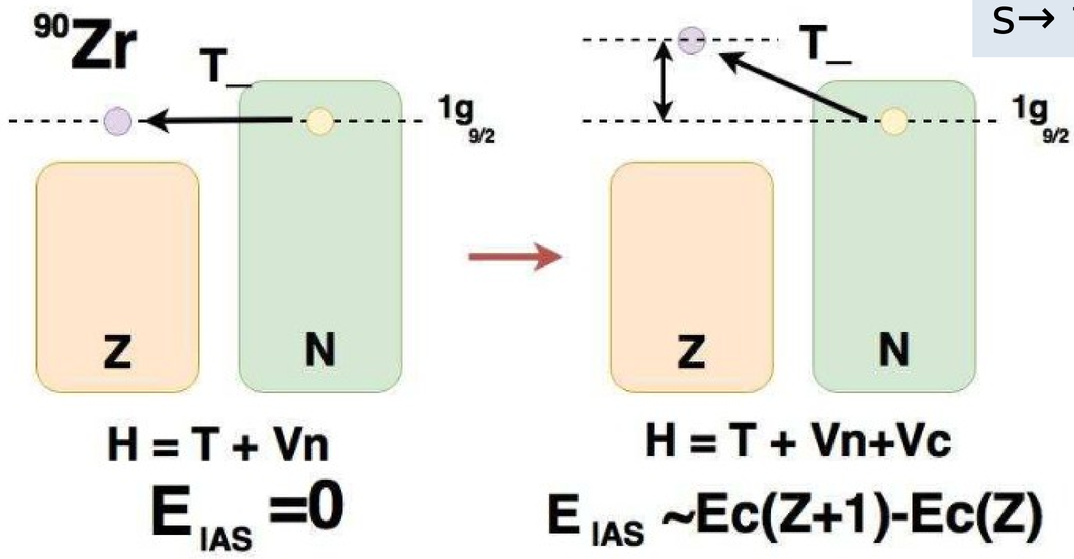
$$a_{\rm sym}(A) = \frac{J}{1+x_A}, \quad \text{with} \quad x_A = \frac{9J}{4Q}A^{-1/3}. \qquad \Delta r_{np}^{\rm DM} = \frac{2r_0}{3J}[J - a_{\rm sym}(A)]A^{1/3}(I-I_C)$$

$$\alpha_D \approx \frac{A\langle r^2\rangle}{12J}\left[1+\frac{5}{2}\frac{\Delta r_{np} + \sqrt{\frac{3}{5}}\frac{e^2 Z}{70J} - \Delta r_{np}^{\rm surface}}{\langle r^2\rangle^{1/2}(I-I_C)}\right]$$

Polarizability must increase with the mass (for the dipole $A^{5/3}$, for the quadrupole $A^{7/3}$ and so on) and **surface symmetry energy** and **decrease** with the **bulk symmetry energy**

Use of sum rules:
Fermi or Isobaric Analog Resonance

$$F = T_{\pm} = \sum_{i}^{A} t_{\pm}(i)$$



Isospin algebra analogous to spin algebra
s→ t and τ → σ (Pauli matrices with t=τ/2)

$$t_-|n\rangle = \frac{1}{2}|p\rangle$$

$$t_+|p\rangle = -\frac{1}{2}|n\rangle$$

$$T_+^\dagger = T_- \qquad T_-^\dagger = T_+$$

$$[T_z, T_\pm] = \pm T_\pm$$

$$[T_+, T_-] = 2T_z$$

→ non-energy weighted sum rule:

$$m_0^- - m_0^+ = \langle 0|T_+T_-|0\rangle - \langle 0|T_-T_+|0\rangle$$
$$= \langle 0|[T_+, T_-]|0\rangle = \langle 0|2T_z|0\rangle$$
$$= N - Z$$

Note: If not isospin-mixing it would be zero!!

→ energy weighted sum rule:

$$m_1 = \sum_\nu (E_\nu - E_0)|\langle \nu|F|0\rangle|^2 = \langle 0|T_+[\mathcal{H}, T_-]|0\rangle$$

[H,T₋] different from zero only if **H** contains terms that **breaks isospin invariance**

Use of sum rules:
Fermi or Isobaric Analog Resonance

$$F = T_\pm = \sum_i^A t_\pm(i)$$

→ Hence, the **centroid energy** m_1/m_0:

$$E_{\text{IAS}} = \frac{\langle 0|T_+[\mathcal{H},T_-]|0\rangle}{\langle 0|T_+T_-|0\rangle} = \frac{1}{N-Z}\langle 0|T_+[\mathcal{H},T_-]|0\rangle$$

→ Assuming a simple model: **indepenent particle** model with only **Coulomb breaking isospin symmetry** (neglect exchange effects)

$$E_{\text{IAS}}^{\text{C,direct}} = \frac{1}{N-Z}\int\left[\rho_n(\vec{r}) - \rho_p(\vec{r})\right] U_{\text{C}}^{\text{direct}}(\vec{r})d\vec{r},$$

$$U_{\text{C}}^{\text{direct}}(\vec{r}) = \int \frac{e^2}{|\vec{r}_1 - \vec{r}|}\rho_{\text{ch}}(\vec{r}_1)d\vec{r}_1$$

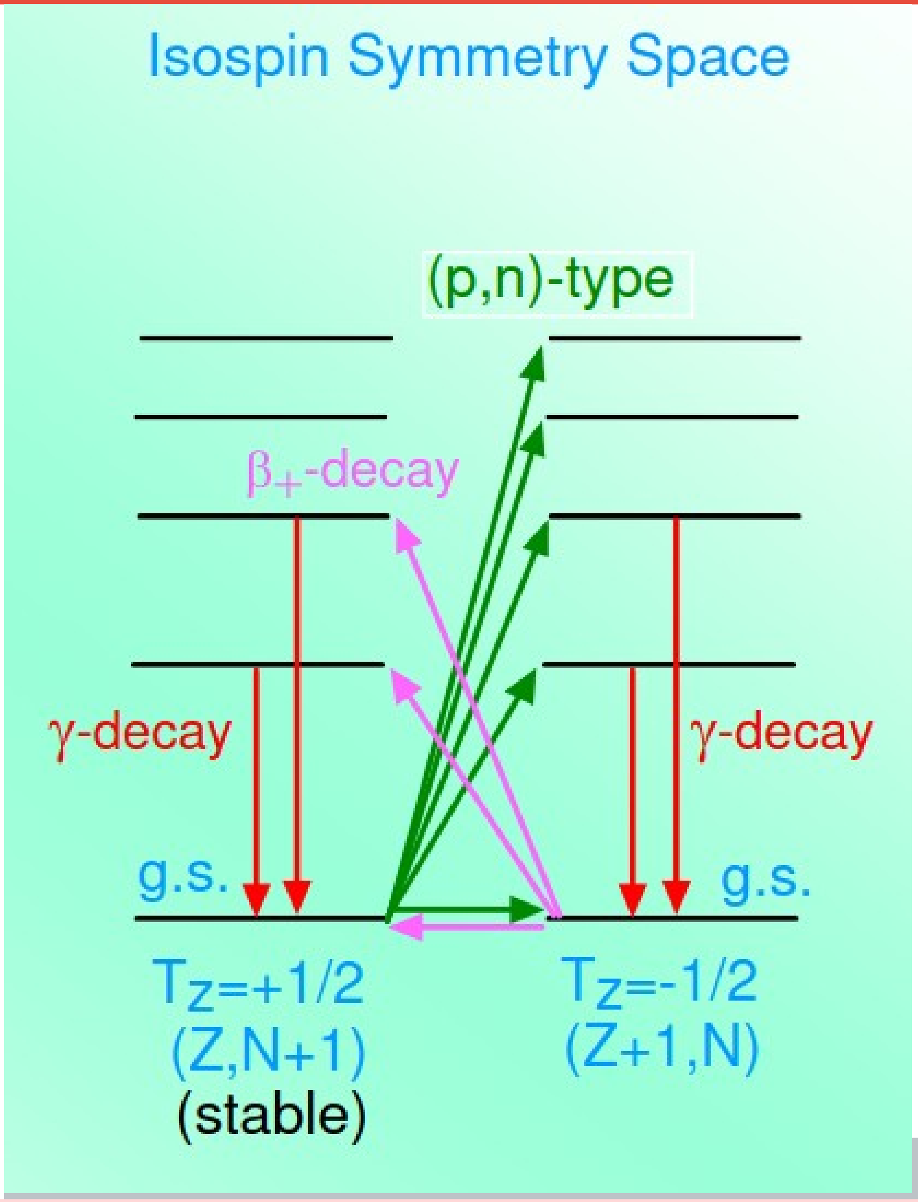
→ Assuming sharp sphere to describe ρ_n and ρ_p and $\rho_{ch} = \rho_p$

$$U_{\text{C}}^{\text{direct}}(\vec{r}) = \begin{cases} \frac{Ze^2}{2R_p}\left(3 - \frac{r^2}{R_p^2}\right) & \text{for } r < R_p \\ \frac{Ze^2}{r} & \text{for } r > R_p \end{cases}$$

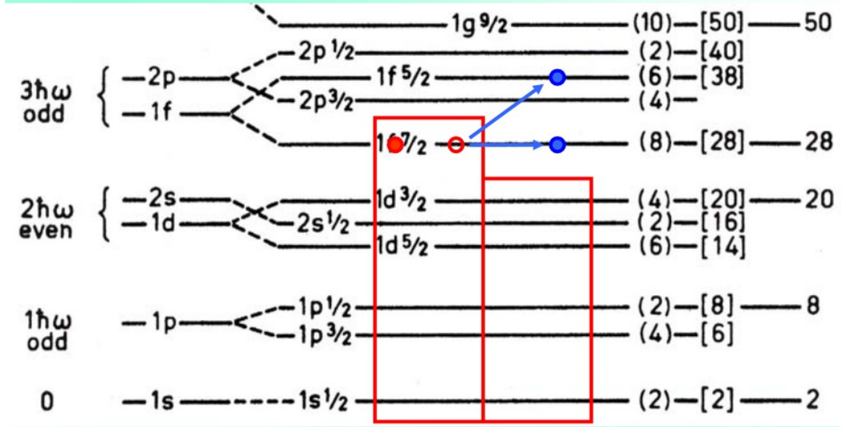
$$\begin{aligned} E_{\text{IAS}} &\approx E_{\text{IAS}}^{\text{C,direct}} \\ &\approx \frac{6}{5}\frac{Ze^2}{R_p}\left(1 - \frac{1}{2}\frac{N}{N-Z}\frac{R_n - R_p}{R_p}\right) \\ &\approx \frac{6}{5}\frac{Ze^2}{r_0A^{1/3}}\left(1 - \sqrt{\frac{5}{12}}\frac{N}{N-Z}\frac{\Delta R_{\text{np}}}{r_0A^{1/3}}\right) \end{aligned}$$

Use of sum rules:
Gamow-Teller Resonance

$$F = O_{\pm}^{\mathrm{GT}} = \sum_{i=1}^{3}\sum_{j=1}^{A}\sigma_i(j)t_{\pm}(j)$$

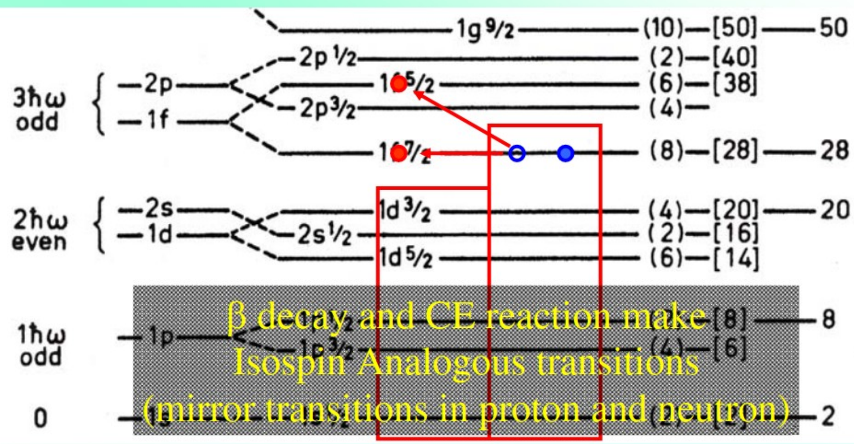


Transitions from ^{42}Ti : β decay



proton: $f_{7/2}$ → neutron $f_{7/2}$
proton: $f_{7/2}$ → neutron $f_{5/2}$

Transitions from ^{42}Ca : CE Reaction



neutron: $f_{7/2}$ → proton $f_{7/2}$
neutron: $f_{7/2}$ → proton $f_{5/2}$

http://www0.mi.infn.it/~colo/lectures/lectures.html

Use of sum rules:
Gamow-Teller Resonance

$$F = O_{\pm}^{\mathrm{GT}} = \sum_{i=1}^{3}\sum_{j=1}^{A}\sigma_i(j)t_{\pm}(j)$$

→ Non-energy weighted sum-rule (m_0):

$$S_- - S_+ = \sum_f |\langle f|O_-^{\mathrm{GT}}|0\rangle|^2 - \sum_f |\langle f|O_+^{\mathrm{GT}}|0\rangle|^2 = \langle 0|[O_+^{\mathrm{GT}}, O_-^{\mathrm{GT}}]|0\rangle = 3(N-Z)$$

sum up three components of the spin operator

→ Centroid energy (m_1/m_0):

A **collective state** could be represented as a **coherent particle-hole superposition** (as we will discuss). The state can be **effectively induced** by a **one-body average field** (proportional to σ · τ operators in the GT case) or **equivalently** by a two-body interaction written in a separable form (Bohr&Mottelson)

spin-orbit, isospin, spin, spin-isospin

$$V = \sum_i^A \kappa_{ls}\boldsymbol{l}(i)\cdot\boldsymbol{s}(i) + \frac{1}{2}\frac{\kappa_\tau}{A}\sum_{i\neq j}^A \boldsymbol{\tau}(i)\cdot\boldsymbol{\tau}(j) + \frac{1}{2}\frac{\kappa_\sigma}{A}\sum_{i\neq j}^A \boldsymbol{\sigma}(i)\cdot\boldsymbol{\sigma}(j) + \frac{1}{2}\frac{\kappa_{\sigma\tau}}{A}\sum_{i\neq j}^A (\boldsymbol{\sigma}(i)\cdot\boldsymbol{\sigma}(j))(\boldsymbol{\tau}(i)\cdot\boldsymbol{\tau}(j)),$$

$$E_{\mathrm{GT}} - E_{\mathrm{IAS}} = \frac{\langle 0|[O_+, [V, O_-]]|0\rangle}{(N-Z)} = -\frac{4}{3}\frac{\kappa_{ls}}{N-Z}\langle 0|\sum_i \boldsymbol{l}(i)\cdot\boldsymbol{s}(i)|0\rangle + 2(\kappa_{\sigma\tau}-\kappa_\tau)\frac{N-Z}{A}.$$

Use of sum rules:
Spin Dipole Resonance

$$\sum_{i=1}^{A} \sum_{M} t_{\pm}(i) r_i^L \left[Y_L(\hat{r}_i) \otimes \boldsymbol{\sigma}(i) \right]_{JM}$$

→ Non-energy weighted sum-rule (m_0):

$$m_0(t_-) - m_0(t_+) = \langle 0|O_-^{\mathrm{SD}}|0\rangle - \langle 0|O_+^{\mathrm{SD}}|0\rangle$$

$$= \langle 0|[O_-^{\mathrm{SD}}, O_+^{\mathrm{SD}}]|0\rangle = \frac{9}{4\pi} \sum_{i=1}^{A} \langle 0|r_i^2 [t_-(i), t_+(i)]|0\rangle$$

$$= 2\frac{9}{4\pi} \sum_{i=1}^{A} \langle 0|r_i^2 t_z(i)|0\rangle = \boxed{\frac{9}{4\pi}\left(N\langle r_n^2\rangle - Z\langle r_p^2\rangle\right)}$$

→ Rewritting it in terms of the neutron skin thickness: $\Delta r_{np} = \langle r_n^2\rangle^{1/2} - \langle r_p^2\rangle^{1/2}$

$$m_0(t_-) - m_0(t_+) = \frac{9}{4\pi}(N-Z)\langle r_p^2\rangle \left[1 + \frac{2N}{N-Z}\frac{\Delta r_{np}}{\langle r_p^2\rangle^{1/2}} + \frac{N}{N-Z}\left(\frac{\Delta r_{np}}{\langle r_p^2\rangle^{1/2}}\right)^2\right]$$

$$\approx \frac{9}{4\pi}(N-Z)\langle r_p^2\rangle \left(1 + \frac{2N}{N-Z}\frac{\Delta r_{np}}{\langle r_p^2\rangle^{1/2}}\right)$$

The only straight forward sum rule giving information on the skin!!